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On Some Fixed Point Results In Complete B_2 – Metric Space

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Abstract:In this paper,obtain fixed point results for single and multi-valued mappings in the structure of extended b_2 - metric space. Our results extend the results of Kiran et al. [11] and others. Moreover, an example is given at the end to show the superiority of our results.

Keywords- b - metric space, extended b- metric space, 2- metric space, b_2 – metric space, extended b_2 – metric space.

1. INTRODUCTION

In many branches of sciences, economics, computer science, engineering and the development of non- linear analysis, the fixed point theory is one of the most important tool. In 1989, Bakhtin [4] introduced the concept of b- metric space and established many results and others (see [1],[2], [3], [5], [14]& [15]). In 2017, Kamran [10] introduced the concept of extended b- metric space and established many results and others (see [17], [18]).

On the other hand in 1963, Gähler[8] introduced the concept of 2-metric having the area of triangle of R^3as the imperative example. Similarly, several fixed point results were obtained for mapping in such space and many other authors (see [7], [9],[12], [15]&[16]). In 2014, Parvaneh al et. [14]introducedb_2 - metric space and established many fixed point results and many other authors (see [5]&[10]).

In this paper,obtain fixed point results for single and multi-valued mappings in the structure of extended b_2 - metric space. Our results extend the results of Kiran et al. [11] and others. Moreover, an example is given at the end to show the superiority of our results.

2. PRELIMINARIES

Definition 2.1 [6] Let X be a set and $s \geq 1$ a real number. A function $d : X \times X \rightarrow [0, \infty)$ is called a b - metric space, if it satisfied the following axioms for all x, y, z ε X,

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then pair (X, d) is called a b - metric space with parameter s.

Example 2.2[6] Let (X, d) be a metric space and let $\beta \geq 1$, $\lambda \geq 0$ and $\mu > 0$ for x, yε X. Set

$\rho(x, y) = \lambda d(x, y) + \mu d(x, y)^{\beta}$. Then (X, ρ) is a b - metric space with the parameter $s = 2^{\beta-1}$ and not a metric space on X.

Definition 2.3 [10]Let X be a non- empty set and $\varphi : X \times X \to [1, \infty)$. A function $d_\varphi : X \times X \to [0, \infty)$ is called an extended b - metric space, if for all x, y, z ε X, it satisfies.

1. $d_\varphi(x, y) = 0$ if and only if x = y,2. $d_\varphi(x, y) = d_\varphi(y, x)$,3. $d_\varphi(x, y) \leq \varphi(x, y)[d_\varphi(x, z) + d_\varphi(z, y)]$
The pair (X, d_φ) is called an extended b – metric space.

Example2.4 [10]Let X = [0, ∞). Define $d_\varphi : X \times X \to [0, \infty)$ by

$$d_\varphi(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 3, & \text{if } x \text{ or } y \in \{1, 2\}, x \neq y, \\ 5, & \text{if } x \neq y \in \{1, 2\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, d_φ) is an extended b - metric space, where $\varphi : X \times X \to [1, \infty)$is defined by $\varphi(x, y) = x + y + 1$ for all x, y ε X.

Remark 2.5 [10] Every b - metric space is an extended b - metric space with constant function $\varphi(x, y) = s$ if $s \geq 1$ but its converse is not true in general.

Lemma 2.6 [11] Every sequence $\{x_n\}_{n \in N}$ if elements from an extended b - metric space (X, d_φ) having the property that for every n ε N, there exists μ ε [0,1) such that

$d_\varphi(x_{n+1}, x_n) \leq \mu\, d_\varphi(x_n, x_{n-1})$,

where for each x_0 ε X, $\lim_{n,m \to \infty} \varphi(x_n, x_m) < 1/\mu$. Then $\{x_n\}^{\infty}_{n=0}$ is a Cauchy sequence.

Lemma 2.7 [11] Every sequence $\{x_n\}_{n \in N}$if elements from an extended b - metric space (X, d_φ), the inequality

$d_\varphi(x_0, x_k) \leq \sum_{i=0}^{k-1} d_\phi(x_i, x_i + 1) \prod_{l=0}^{i} \phi(x_l, x_k)$ is valid for every k ε N.

Definition 2.8 [10] Let X be a non-empty set and let function $d : X \times X \times X \to [0,\infty)$ be a mapping satisfying

1. For every pair of distinct points x, y ε X, three exists a point z ε X such that $d(x, y, z) \neq 0$ 2. If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$ 3. $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all x, y, z εX4. $d(x, y, z) \leq d(x, z, t) + d(x, t, z) + d(t, y, z)$ for all x, y, z, t ε X.

Then d is called a 2 – metric on X and (X, d) is called a 2 – metric space.

Definition 2.9 [14] Let X be a non- empty set X. $s \geq 1$ be a number and let function $d : X \times X \times X \to [0,\infty)$ be a mapping satisfying the following conditions.

1. For every pair of distinct points x, y ε X, three exists a point z ε X such that $d(x, y, z) \neq 0$,2. If at least two of three points x, y, z are the same, the $d(x, y, z) = 0$, 3. $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all x, y, z ε X, 4. $d(x, y, z) \leq s\,[d(x, z, t) + d(x, t, z) + + d(t, y, z)]$ for all x, y, z, t ε X.

Then (X, d) is called a b_2 - metric space with parameter s.

Remark 2.10 for s = 1, b_2 - metric space reduces to 2 – metric space.

Example 2.11 [14] Let X = [0, ∞). Define the function $d : X \times X \times X \to [0, \infty)$ by $d(x, y, z) = (xy + yz + zx)^p$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$, where $p \geq 1$ is a real number. Evidently, from convexity of function $f(x) = x^p$ for $x \geq 0$, then by Jensen inequality we have

$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p)$

So, one can obtain the result that (X, d) is a b_2 - metric space with $s \leq 3^{p-1}$.

Definition 2.12 Let X be a non- empty set and $\varphi : X \times X \times X \to [1, \infty)$. A function $d_\varphi : X \times X \times X \to [0, \infty)$ is called an extended b_2 - metric space, if for all x, y, z, t ε X, it satisfies.

1. For every pair of distinct points x, y ε X, three exists a point z ε X such that $d_\varphi(x, y, z) \neq 0$,
2. If at least two of three points x, y, z are the same, the $d_\varphi(x, y, z) = 0$,
3. $d_\varphi (x, y, z) = d_\varphi (x, z, y) = d_\varphi (y, x, z) = d_\varphi (y, z, x) = d_\varphi (z, x, y) = d_\varphi (z, y, x)$ for all x, y, z ε X,
4. $d_\varphi (x, y, z) \leq \varphi (x, y, z) [d_\varphi (x, z, t) + d_\varphi (x, t, z) + d_\varphi (t, y, z)]$ for all x, y, z, t ε X. Then (X, d_φ) is called an extended b_2 - metric space.

Example 2.13 Let $X = \{ (\alpha, 0) : \alpha = 1/2, 1/2^2, \ldots, 1/2^n, \ldots \} \cup \{ 0, 1 \} C R^3$ and letd_φ (x, y, z) denote the sequenceof the area of triangle with vertex x, y, z ε X. $d_\varphi ((\alpha, 0) ,(\beta, 0) ,(0, 1)) = (\alpha - \beta)^2/4$, $\phi ((\alpha, 0) ,(\beta, 0) ,(0, 1)) = \alpha + \beta +1$. Then X is an extended$b_2$ - metric space.

Remark 2.14 Every b_2 – metric space is an extended b_2 – metric space with constant function $\varphi (x, y, z) = s$ if $s \geq 1$, but its converse is not true in general.

Definition 2.15 [1] Let (X, d_ϕ) be an extended b_2 – metric space, where$\varphi : X \times X \times X \to [1, \infty)$ is bounded. Then for all A, B, C εC B(X) denotes the family of all non-empty closed and bounded subset of X, the Hausdorff- Pompieu metric on C B(X) induced by d_ϕ is defined by

$$H_\phi (A, B, C) = \max \{ \sup_{a \varepsilon A} d_\phi(a, B, C), \sup_{b \varepsilon B} d_\phi(b, C, A), \sup_{c \varepsilon C} d_\phi(c, A, B)\}$$

Where for every a ε A

$$d_\phi(a, B, C) = \inf \{ d_\phi(a, b, c) : b \varepsilon B, c \varepsilon C \}$$ and

$\phi : C B(X)\times C B(X) \times C B(X) \to [1, \infty)$ is such that $\phi (A, B, C) = \sup \{ \phi (a, b, c) ; a \varepsilon A, b \varepsilon B, c \varepsilon C \}$.

Definition 2.16[11] Let X be any set. A function T : X→ C B(X) be a multi- valued map. For any point x_0 ε X, the sequence $\{x_n\}_{n=0}^{\infty}$given by $x_{n+1} \varepsilon Tx_n$, n = 0, 1, 2, … is called an iterative sequence with initial point x_0.

Definition 2.17 [11] Let (X, d_ϕ) be an extended b – metric space. A function T : X→ C B(X) is called continuous , if for every sequence $\{x_n\}_{n \varepsilon N}$ and $\{y_n\}_{n \varepsilon N}$ belongs to X and x, y ε X such that

$$\lim_{n\to\infty}x_n = x, \quad \lim_{n\to\infty}y_n = y \text{ and } y_n \varepsilon Tx_n. \text{ We have } y \varepsilon Tx.$$

Definition 2.18[11] An extended b – metric space(X, d_ϕ) is called ẋ - continuous, it for every A ε C B(X),

$\{x_n\}_{n\varepsilon N} \varepsilon$ X and xε X such that $\lim_{n\to\infty}x_n = x$. We have $\lim_{n\to\infty}d_\phi(x_n, A) = d_\phi(x, A)$.

Remark 2.19 [8] Note that ẋ - continuous of d_ϕ is stronger than continuity of d_ϕ in first variable.

In [11], the author introduced the following results, which improve the results of [20], [23]

.**Theorem 2.20 [11]**Let (X, d_φ) be a complete extended b – metric space with $\varphi : X\times X\to[1, \infty)$.

If T : X → X satisfies the inequality

$$d_\varphi (Tx, T y)\leq k_1 d_\varphi (x, y) + k_2 d_\varphi (x, Tx) + k_3 d_\varphi (y, Ty) + k_4 [d_\varphi (y, Tx) +d_\varphi (x, Ty)],$$

where $k_i \geq 0$, for I = 1, 2, 3, 4 and for each $x_0 \varepsilon$ X,

$k_1+ k_2 +k_3 +2k_4 \lim_{n,m\to\infty}\phi (x_n, x_m) < 1$, then T has a fixed point.

In [11], the author introduced the following results, which improve the results of [20],

Theorem 2.21 [11]Let (X, d_φ) be a complete extended b – metric space with $\varphi : X\times X\to[1,\infty)$.

If $T : X \to X$ satisfies the inequality

$d_\varphi(Tx, Ty) \leq k_1 d_\varphi(x, y) + k_2[d_\varphi(x, Tx) + d_\varphi(y, Ty)]$, for each x, y ε X, where k_1, k_2 ε [0, 1/3), Moreover for each x_0ε X,

$k_2 \lim_{n,m\to\infty}\phi(x_n, x_m) < 1$, then T has a unique fixed point.

Theorem 2.22 [11]Let (X, d_φ) be a complete extended b – metric space. LetT : X→ C B(X) be a multi- valued mapping having the property that there exists c_1, c_2ε [0,1] and ηε[0,1] such that

(i) For each x_0ε X, $\lim_{n,m\to\infty}\eta\, c_2\phi(x_n, x_m) < 1$, here $x_n = T^n x_0$,

(ii) $H_\phi(Tx, Ty) \leq \eta\, N_{c_1,c_2}(x, y)$ for all x, y ε X.

Then for every x_0ε X, there exists με [0,1] and a sequence $\{x_n\}_{n\varepsilon N}$ of iterates from X such that for every n ε N, $d_\phi(x_n, x_{n+1}) \leq \mu\, d_\phi(x_{n-1}, x_n)$.

Theorem 2.23 [11]Let (X, d_φ) be a complete extended b – metric space. LetT : X→ C B(X) be a multi- valued mapping having the property that there exists c_1, c_2ε [0,1] and ηε [0,1] such that

(i) For each x_0ε X, $\lim_{n,m\to\infty}\eta\, c_2\phi(x_n, x_m) < 1$, here $x_n = T^n x_0$,

(ii) $H_\phi(Tx, Ty) \leq \eta\, N_{c_1,c_2}(x, y)$ for all x, y ε X.

(iii) T is continuous.

Then T has a fixed point in X.

Theorem 2.24 [11]Let (X, d_φ) be a complete extended b_2 - metric space. LetT : X→ C B(X) be a multi- valued mapping having the property that there exists c_1, c_2ε [0,1] and ηε [0,1] such that

(i) For each x_0ε X, $\lim_{n,m\to\infty}\eta\, c_2\phi(x_n, x_m) < 1$, here $x_n = T^n x_0$,

(ii) $H_\phi(Tx, Ty) \leq \eta\, N_{c_1,c_2}(x, y)$ for all x, y ε X.

(iii) T is ϰ - continuous.

Then T has a fixed point in X.

Theorem 2.25[11] A multi – valued mapping T :X→ C B(X) has a fixed point in a complete extended b_2 – metric space (X, d_φ) if it satisfies the following two axioms

(i) There exists c_1, c_2ε [0,1] and ηε [0,1] such that $H_\phi(Tx, Ty) \leq \eta\, N_{c_1,c_2}(x, y)$ for all x, y ε X,

(ii) For each x_0ε X, $\max\{\eta\, c_1 \lim_{n,m\to\infty}\phi(x_n, x_m), \eta\, c_2 \lim_{n,m\to\infty}\phi(x_n, x_m)\} < 1$, here $x_n = T^n x_0$.

3. MAIN RESULTS

Definition 3.1 Let (X, d_ϕ) be an extended b_2 – metric space. A function T : X→ C B(X) is called continuous , if for every sequence $\{x_n\}_{n\varepsilon N}$ and $\{y_n\}_{n\varepsilon N}$ belongs to X and x, y ε X such that

$\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and y_n ε Tx_n. We have y ε Tx.

Definition 3.2 An extended b_2 – metric space(X, d_ϕ) is called ϰ - continuous, it for every A, Bε C B(X),

$\{x_n\}_{n\varepsilon N}$ε X and xε X such that $\lim_{n\to\infty} x_n = x$. We have $\lim_{n\to\infty} d_\phi(x_n, A, B) = d_\phi(x, A, B)$.

Lemma 3.3 $d_\varphi(x_n, x_{n-1}, x_{n+1}) = 0$.

Lemma 3.4 Every sequence $\{x_n\}_{n \varepsilon N}$ if elements from an extended b_2 – metric space (X, d_φ), the inequality

$d_\varphi(x_0, x_k, a) \leq \sum_{i=0}^{k-1} d_\phi(x_i, x_i + 1, a) \prod_{l=0}^{i} \phi(x_l, x_k, a)$ is valid for every k ε N.

Proof- $d_\varphi(x_0, x_k, a) \leq \varphi(x_0, x_k, a)[d_\varphi(x_0, x_k, x_1) + d_\varphi(x_0, x_1, a) + d_\varphi(x_1, x_k, a)]$. Then by lemma 3.3,

$d_\varphi(x_0, x_k, a) \leq \varphi(x_0, x_k, a) d_\varphi(x_0, x_1, a) + \varphi(x_0, x_k, a)\varphi(x_1, x_k, a) d_\varphi(x_1, x_2, a) + \ldots + \varphi(x_0, x_k, a)\varphi(x_1, x_k, a)\ldots\varphi(x_{k-1}, x_k, a) d_\varphi(x_{k-1}, x_k, a)$.

This implies that $d_\varphi(x_0, x_k, a) \leq \sum_{i=0}^{k-1} d_\phi(x_i, x_i + 1, a) \prod_{l=0}^{i} \phi(x_l, x_k, a)$.

Lemma 3.5 Every sequence $\{x_n\}_{n\varepsilon N}$ if elements from an extended b_2 – metric space (X, d_φ), having the property that for every n ε N, there exists μ ε [0,1) such that

$d_\varphi(x_{n+1}, x_n, a) \leq \mu\, d_\varphi(x_n, x_{n-1}, a)$. Then $\{x_n\}$ is a Cauchy sequence. ……………… 3.1

Proof- First by successively applying **3.1**, we get

$d_\varphi(x_{n+1}, x_n, a) \leq \mu^n d_\varphi(x_1, x_0, a)$ …………………………… 3.2

for every n ε N. Then by the lemma3.4, for all m. k ε N, we have

$d_\varphi(x_m, x_{m+k}, a) \leq \sum_{n=m}^{m+k-1} d_\phi(x_n, x_n + 1, a) \prod_{l=0}^{n} \phi(x_l, x_m + k, a)$

$\leq d_\phi(x_0, x_1, a) \sum_{n=0}^{k-1} \mu^n \prod_{l=0}^{n} \phi(x_l, x_m + k, a)$

$\leq d_\phi(x_0, x_1, a) \sum_{n=0}^{k-1} \mu^{n+m} \prod_{l=0}^{n+m} \phi(x_l, x_m + k, a)$

$\leq \mu^m d_\phi(x_0, x_1, a) \sum_{n=0}^{k-1} \mu^n \prod_{l=0}^{n+m} \phi(x_l, x_m + k, a)$

$\leq \mu^m d_\phi(x_0, x_1, a) \sum_{n=0}^{k-1} \mu^{log_\mu \prod_{l=0}^{n+m} \phi(x_l, x_m + k, a) + n}$. …… **3.3**

Now let us take two case for

$\log_\mu \prod_{l=0}^{n+m} \phi(x_l, x_m + k, a) + n$.

Case I. If $\prod_{l=0}^{n+m} \phi(x_l, x_m + k, a)$ is finite, let us say M, then $\lim_{n\to\infty} \log_\mu M + n = \infty$.

Hence the series $\sum_{n=0}^{k-1} \mu^{log_\mu M + n}$is convergent.

Case II. If $\prod_{l=0}^{n+m} \phi(x_l, x_m + k, a)$ is infinite, then $\lim_{n\to\infty} \log_\mu \prod_{l=0}^{n+m} \phi(x_l, x_m + k, a) = 0$, so there exist n_0 ε N such that $\mu^{\log_\mu \prod_{l=o}^{n+m} \phi(x_l, x_m + k, a)} > M$, ie

$\mu^{\log_\mu \prod_{l=o}^{n+m} \phi(x_l, x_m + k, a) + n} \leq \mu^M \mu^n$, for each n ε N, $n \geq n_0$.

Hence the series $\sum_{n=0}^{k-1} \mu^{log_\mu \prod_{l=0}^{n+m} \phi(x_l, x_m + k, a) + n}$ is convergent. If both cases denoting by

S the sum of this series. We come to the conclusion that

$d_\varphi(x_m, x_{m+1}, a) \leq \mu^m d_\varphi(x_1, x_0, a)s$, for all m, k ε N.

Consequently, as $\lim_{n\to\infty} \mu^m = 0$, we conclude that $\{x_m\}$mεN is a Cauchy sequence.

Lemma 3.6 Let A, B, C ε CB(X) , then for every μ > 0 and b ε B, c ε C there exists a ε A such that

$d_\varphi(a, b, c) \leq H_\varphi(A, B, C) + \mu$. …………………… 3.4

Proof-By definition of Hausdorff 2- metric, for A, B, C ε CB(X) and for b ε B, c ε C, we have

$d_\varphi(a, b, c) \leq H_\varphi(A, B, C)$.

By the definition of infimum, we can let $\{a_n\}$ be a sequence in A such that

$d_\varphi(b, a_n, c) \leq H_\varphi(b, a, C) + \mu$, where $\mu > 0$. ………………… 3.5

We know that A is closed and bounded, so there exists a ε A such that $a_n \to a$.

Therefore, by 3.5, we have $d_\varphi(a, b, c) \leq d_\varphi(A, b, c) + \mu \leq H_\varphi(A, B, C) + \mu$.

Theorem 3.7 Let (X, d_φ) be a complete extended b_2 – metric space with $\varphi : X \times X \times X \rightarrow [1, \infty)$.

If $T : X \rightarrow X$ satisfies the inequality

$d_\varphi (Tx, T y, a) \leq k_1 d_\varphi (x, y, a) + k_2 d_\varphi (x, Tx, a) + k_3 d_\varphi (y,Ty, a) + k_4 [d_\varphi (y, Tx, a) +d_\varphi (x, Ty,a)$, ….3.6

where $k_i \geq 0$, for I = 1, 2, 3, 4 and for each $x_0 \varepsilon X$,

$k_1+ k_2 +k_3 +2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 1$, then T has a unique fixed point.

Proof-Let us choose an arbitrary $x_0 \varepsilon X$ and define the iterative sequence $\{x_n\}_{n=0}^{\infty}$ by $x_n = T^{n-1}x_0$ for all $n \geq 1$.

If $x_n = x_{n-1}$, then x_n is a fixed point of T the proof holds. So suppose $x_n \neq x_{n-1}$ for all $n \geq 1$. Then from equation 3.6, we have

$d_\varphi (Tx_n, T x_{n-1}, a) \leq k_1 d_\varphi (x_n, x_{n-1}, a)+k_2 d_\varphi(x_n, Tx_n, a)+k_3 d_\varphi (x_{n-1}, Tx_{n-1}, a)+ k_4 [d_\varphi (x_n, x_n, a)+d_\varphi (x_{n-1}, x_{n+1},a)]$

$= k_1 d_\varphi (x_n, x_{n-1}, a)+k_2 d_\varphi(x_n, x_{n+1}, a)+k_3 d_\varphi (x_{n-1}, x_n, a)+ k_4 [d_\varphi (x_n, x_n, a)+d_\varphi (x_{n-1}, x_{n+1},a)]$.

By triangle inequality, we have

$d_\varphi (Tx_n, T x_{n-1}, a) \leq k_1 d_\varphi (x_n, x_{n-1}, a)+k_2 d_\varphi(x_n, x_{n+1}, a)+k_3 d_\varphi (x_{n-1}, x_n, a)+ k_4 \varphi (x_{n-1}, x_{n+1}, a) [d_\varphi (x_{n-1}, x_{n+1}, x_n)+ d_\varphi (x_{n-1}, x_n,a)+d_\varphi (x_n, x_{n+1},a)]$

$= k_1 d_\varphi (x_n, x_{n-1}, a)+k_2 d_\varphi(x_n, x_{n+1}, a)+k_3 d_\varphi (x_{n-1}, x_n, a)+ k_4 \varphi (x_{n-1}, x_{n+1}, a) [d_\varphi (x_{n-1}, x_{n+1},a) + d_\varphi (x_{n-1}, x_{n+1},a)]$, by lemma 3.3

$=[k_1 + k_3+ k_4 \varphi (x_{n-1}, x_{n+1}, a)] d_\varphi (x_{n-1}, x_{n+1},a) +[k_2+ k_4 \varphi (x_{n-1}, x_{n+1}, a)]d_\varphi (x_n, x_{n+1},a)$ …………… 3.7

Similarly,

$d_\varphi (x_n, x_{n+1}, a) \leq [k_1 + k_2 + k_4 \varphi (x_{n-1}, x_{n+1}, a)] d_\varphi (x_n, x_{n-1},a) +[k_3+ k_4 \varphi (x_{n-1}, x_{n+1}, a)] d_\varphi (x_n, x_{n+1},a)$. … 3.8

By adding equation 3.7 and 3.8, we have

$d_\varphi (x_{n+1},x_n, a) \leq \mu\, d_\varphi (x_n, x_{n-1}, a)$

where $\mu = [2k_1 + k_2 + k_3 + k_4 \varphi (x_{n-1}, x_{n+1}, a)] /[2 - k_2 – k_3- k_4 \varphi (x_{n-1}, x_{n+1}, a)]$.

Since $k_1 + k_2 +k_3 +2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 1$,

Or, $2k_1 + 2k_2 +2k_3 +4k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 2$,

Or, $2k_1+ k_2 +k_3 +2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 2 - k_2 -k_3 -2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a)$

Or, $2k_1 + k_2 +k_3 +2k_4 \lim_{n,m\rightarrow\infty} c /2 - k_2 -k_3 -2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 1$

Or, $\mu < 1$.

Hence from lemma 3.5, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As X is complete, therefore there exists $x \varepsilon X$ such that $\lim_{n\rightarrow\infty} x_n = x$. Next , we show that x is a fixed point of T.

From the triangle inequality and equation 3.6, we have

$d_\varphi (x, Tx, a) \leq \phi (x, Tx, a) [d_\varphi (x, Tx, x_{n+1})+ d_\varphi (x, x_{n+1}, a) +d_\varphi (x_{n+1}, Tx, a)]$, by lemma 3.3

$\leq \phi (x, Tx, a) [d_\varphi (x, x_{n+1}, a) +d_\varphi (x_{n+1}, Tx, a)] \leq \phi (x, Tx, a) [d_\varphi (x, x_{n+1}, a) +k_1 d_\varphi (x_n, x, a)+k_2 d_\varphi(x_n, x_{n+1}, a)+k_3 d_\varphi (x, Tx, a)+ k_4 [d_\varphi (x_n, Tx, a)+d_\varphi (x, x_{n+1},a)]]$,

$\leq \phi (x, Tx, a) [d_\varphi (x, x_{n+1}, a) +k_1 d_\varphi (x_n, x, a)+k_2 d_\varphi(x_n, x_{n+1}, a)+k_3 d_\varphi (x, Tx, a)+ k_4 d_\varphi (x_n, x_{n+1}, a)+k_4 \phi (x_n, Tx, a) [d_\varphi (x_n, x, a) +d_\varphi (x,Tx, a)]]$

$\leq \phi (x, Tx, a) [(1+k_4)d_\varphi (x, x_{n+1}, a) +(k_1+ k_4 \phi (x_n, Tx, a)) d_\varphi (x,x_n, a) + k_2 d_\varphi(x_n, x_{n+1}, a)+ (k_3 +k_4 \phi (x_n, Tx, a)]d_\varphi (x,T x, a)$.

$[1 - k_3 - k_4 \phi (x_n, Tx, a)] d_\varphi (x,Tx, a) \leq \phi (x, Tx, a)[(1+k_4) d_\varphi (x, x_{n+1}, a) +(k_1+ k_4\phi (x_n, Tx, a)) d_\varphi (x,x_n, a) + k_2 d_\varphi(x_n, x_{n+1}, a)]$. …………………….. 3.9

Similarly,

$[1 - k_2 - k_4 \phi (x_n, Tx, a)] d_\varphi (x,Tx, a) \leq \phi (x, Tx, a)[(1+k_4) d_\varphi (x, x_{n+1}, a) +(k_1+ k_4\phi (x_n, Tx, a)) d_\varphi (x,x_n, a) + k_3 d_\varphi(x_n, x_{n+1}, a)]$. …………………….. 3.10

By adding 3.9 and 3.10, we have,
$[2 - k_2 - k_3 - 2k_4 \phi (x_n, Tx, a)] d_\varphi (x,Tx, a) \leq \phi (x, Tx, a)[2(1+k_4) d_\varphi (x, x_{n+1}, a) +2(k_1+ k_4\phi (x_n, Tx, a)) d_\varphi (x,x_n, a) +(k_2+ k_3) d\varphi(x_n, x_{n+1}, a)] \rightarrow 0$, as $n \rightarrow \infty$. This implies that

$[2 - k_2 - k_3 - 2k_4 \phi (x_n, Tx, a)] d_\varphi (x,Tx, a) \leq 0$.

Since $[2 - k_2 - k_3 - 2k_4 \phi (x_n, Tx, a)] > 0$, we get

$d_\varphi (x,Tx, a) = 0$ implies $x = Tx$.

Now, we show that x is the unique fixed point of T. Assume that x^* is another fixed point of T, then we have to prove that $Tx^* = x^*$.

$d_\varphi (x, x^*, a) = d_\varphi (Tx, Tx^*, a) \leq k_1 d_\varphi (x, x^*, a) + k_2 d_\varphi (x, Tx, a) + k_3 d_\varphi (x^*,Tx^*, a) + k_4 [d_\varphi (x, Tx^*, a) + d_\varphi (x^*, Tx,a)]$

$\leq k_1 d_\varphi (x, x^*, a) + k_2 d_\varphi (x, Tx, a) + k_3 d_\varphi (x^*,Tx^*, a) + k_4 [d_\varphi (x, x^*, a) + d_\varphi (x^*, x,a)]$

$\leq (k_1 + 2k_4) d\varphi (x, x^*, a)$

Implies $(1-k_1 - 2k_4) d\varphi (x, x^*, a) \leq 0$.

$k_1+ k_2 +k_3 +2k_4 \leq k_1 + k_2 +k_3 +2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 1$

implies $(1-k_1 - 2k_4) > 0$, iex $= x^*$. Hence T has a unique fixed point of X.

Remark 3.8 From the symmetry of the distance function d_φ, it is easy to prove similar to that done in [8] that $k_2 = k_3$, thus the inequality 3.6 is equivalent to the following inequality

$d_\varphi (Tx, Ty, a) \leq k_1 d_\varphi (x, y, a) + k_2[d_\varphi (x, Tx, a) + d_\varphi (y,Ty, a)]+ k_4 [d_\varphi (y, Tx, a) + d_\varphi (x, Ty,a)$, ….. 3.11

where $k_i \geq 0$, for I = 1, 2, 4 and for each $x_0 \varepsilon X$,

$k_1,+ 2k_2 +2k_4 \lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) < 1$.

If $k_1 = k_2 = 0$ and $k_4 \varepsilon [0, ½)$ in inequality 3.11, we obtain generalization of [11] in extended b_2 – metric space.

Theorem 3.9 Let (X, d_φ) be a complete extended b_2 – metric space with $\varphi : X \times X \times X \rightarrow [1, \infty)$.

If $T : X \rightarrow X$ satisfies the inequality

$d_\varphi (Tx, Ty, a) \leq k_1 d_\varphi (x, y, a) + k_2 [d_\varphi (x, Tx, a) + d_\varphi (y,Ty, a)]$, ….. 3.12 for each $x, y \varepsilon X$, where $k_1, k_2 \varepsilon [0, 1/3)$. Moreover, for each $x_0 \varepsilon X$,

$\lim_{n,m\rightarrow\infty} \phi (x_n, x_m, a) k_2 < 1$, then T has a unique fixed point.

Proof-Let us choose an arbitrary $x_0 \varepsilon X$ and define the iterative sequence $\{x_n\}_{n=0}^{\infty}$ by $x_n = T^{n-1}x_0$ for all $n \geq 1$.

If $x_n = x_{n-1}$, then x_n is a fixed point of T the proof holds. So suppose $x_n \neq x_{n-1}$ for all $n \geq 1$. Then from equation 3.12, we h ave$d_\varphi (Tx_n, Tx_{n-1}, a) \leq k_1 d_\varphi (x_n, x_{n-1}, a) + k_2 [d_\varphi (x_n, Tx_n, a) + d_\varphi (x_{n-1},Tx_{n-1}, a)]$

$= K_1 d_\varphi (x_n, x_{n-1}, a) + k_2 [d_\varphi (x_n, x_{n+1}, a) + d_\varphi (x_{n-1},x_n, a)]$

$(1 - k_2) d_\varphi (x_n, x_{n+1}, a) \leq (k_1 +k_4) d_\varphi (x_n, x_{n-1}, a)$

$d_\varphi (x_n, x_{n+1}, a) \leq (k_1 +k_4) d_\varphi (x_n, x_{n-1}, a) /(1 - k_2)$

$= \mu d_\varphi (x_n, x_{n-1}, a)$.

Where$\mu = (k_1 +k_4)/(1 - k_2)$. Since $k_1, k_2 \varepsilon [0, 1/3)$, so $\mu < 1$, Hence from lemma 3.5, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As X is complete, therefore there exists x ε X such that $\lim_{n\to\infty} x_n = x$. Next , we show that x is a fixed point of T. From the triangle inequality and equation 3.12, we have

$d_\varphi (x, Tx, a) \leq \phi (x, Tx, a) [d_\varphi (x, Tx, x_{n+1}) + d_\varphi (x, x_{n+1}, a) + d_\varphi (x_{n+1}, Tx, a)]$,

$\leq \phi (x, Tx, a) [d_\varphi (x, x_{n+1}, a) + d_\varphi (x_{n+1}, Tx, a)]$

$\leq \phi (x, Tx, a) [d_\varphi (x, x_{n+1}, a) + k_1 d_\varphi (x_n, x, a) + k_2 [d\varphi(x_n, x_{n+1}, a) + d_\varphi (x, Tx, a)]$.

So, $[1 - k_2 \varphi(x, Tx, a)] d_\varphi (x, Tx, a) \leq 0$, as $n\to\infty$.

Since $\lim_{n,m\to\infty} \phi (x_n, x_m, a) k_2 < 1$,

We have $[1 - k_2\varphi(x, Tx, a)] > 0$, and so $d_\varphi (x, Tx, a) = 0$ ie $x = Tx$

Now, we show that x is the unique fixed point of T. Assume that x^* is another fixed point of T, then we have to prove that $Tx^* = x^*$.

$d_\varphi (x, x^*, a) = d_\varphi (Tx, Tx^*, a) \leq k_1 d_\varphi (x, x^*, a) + k_2 [d_\varphi (x, Tx, a) + d_\varphi (x^*, Tx^*, a)]$

$\leq k_1 d_\varphi (x, x^*, a) < d_\varphi (x, x^*, a)$, which is contradictions. Hence T has a unique fixed point of X.

For x,y ε X and c,d ε [0, 1]. We will use the following notations

$N_{c_1,c_2} (x, y) = \max \{d_\varphi (x, y, a), c_1 d_\varphi (x, Tx, a), c_1 d_\varphi (y, Ty, a), c_2/2[d_\varphi (x, Ty, a) + d_\varphi (y, Tx, a)]$.

Theorem 3.9 Let (X, d_φ) be a complete extended b_2 – metric space. Let$T : X\to CB(X)$ be a multi- valued mapping having the property that there exists $c_1, c_2 \varepsilon [0,1]$ and $\eta \varepsilon [0,1]$ such that

(i) For each $x_0 \varepsilon X$, $\lim_{n,m\to\infty} \eta c_2 \phi (x_n, x_m, a) < 1$, here $x_n = T^n x_0$,

(ii) $H_\phi (Tx, Ty, a) \leq \eta N_{c_1,c_2} (x, y, a)$ for all x, y,aε X.

Then for every $x_0 \varepsilon X$, there exists $\mu \varepsilon [0,1]$ and a sequence $\{x_n\}_{n\varepsilon N}$ of iterates from X such that for every nεN,

$$d_\phi (x_n, x_{n+1}, a) \leq \mu d_\phi (x_{n-1}, x_n, a). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.13$$

Proof- Let us choose an arbitrary $x_0 \varepsilon X$ and $x_1 \varepsilon Tx_0$ consider

$\mu = \max \{\eta, \eta c_2 \phi (x_{n-1}, x_{n+1}, a)/2 - \eta c_2 \phi (x_{n-1}, x_{n+1}, a)\}$.

Clearly, $\mu < 1$. If $x_1 = x_0$, then for every n ε N, the sequence $\{x_n\}_{n\varepsilon N}$ given by $x_n = x_0$ satisfies equation 3.13. Since

$d_\phi (x_1, Tx_1, a) \leq d_\phi (Tx_0, Tx_1, a) \leq H_\phi (Tx_0, Tx_1, a) \leq \eta N_{c_1,c_2} (x_0, x_1, a)$,

there exists $x_2 \varepsilon Tx_1$ such that

$d_\phi (x_1, x_2, a) \leq \eta N_{c_1,c_2} (x_0, x_1, a)$.

If $x_2 = x_1$, then for every n ε N, $n \geq 1$, the sequence $\{x_n\}_{n\varepsilon N}$ given by $x_n = x_1$ satisfies equation 3.13. By repeating thisprocess, we obtained a sequence $\{x_n\}_{n\varepsilon N}$ of elements from X such that $x_{n+1} = Tx_n$ and $0 < d_\phi (x_n, x_{n+1}, a) \leq \eta N_{c_1,c_2} (x_{n-1}, x_n, a)$for every n ε N, $n \geq 1$. Then we have

$0 < d_\phi (x_n, x_{n+1}, a) \leq \eta N_{c_1,c_2} (x_{n-1}, x_n, a)$

$\leq \eta \max \{d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_{n-1}, Tx_{n-1}, a), c_1 d_\varphi (x_n, Tx_n, a), c_2/2[d_\varphi (x_{n-1}, Tx_n, a) + d_\varphi (x_n, Tx_{n-1}, a)\}$.

$\leq \eta \max \{d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_n, x_{n+1}, a), c_2/2[d_\varphi (x_{n-1}, x_{n+1}, a)\}$……………….. 3.14

$\leq \eta \max \{d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_n, x_{n+1}, a), c_2/2 \varphi (x_{n-1}, x_{n+1}, a) \{d_\varphi (x_{n-1}, x_{n+1}, x_n) + d_\varphi (x_{n-1}, x_n, a) + d_\varphi (x_n, x_{n+1}, a)$,

$= \eta \max \{d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_n, x_{n+1}, a), c_2/2\ \varphi (x_{n-1}, x_{n+1}, a) [d_\varphi (x_{n-1}, x_n, a) + d_\varphi (x_n, x_{n+1}, a)]\}$ …………………………… 3.15

For every n ε N.If we take

$\max \{d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_{n-1}, x_n, a), c_1 d_\varphi (x_n, x_{n+1}, a), c_2/2\ \varphi (x_{n-1}, x_{n+1}, a) [d_\varphi (x_{n-1}, x_n, a) + d_\varphi(x_n, x_{n+1}, a)]\}$

$= c_1 d_\varphi (x_n, x_{n+1}, a)$,

Then from equation 3.14and 3.15, $0 < d_\phi (x_n, x_{n+1}, a) \leq \eta c_1 d_\varphi (x_n, x_{n+1}, a) \leq \eta d_\varphi (x_n, x_{n+1}, a)$. As $\eta < 1$. So obtain the contradiction. Therefore, we have

$d_\phi (x_n, x_{n+1}, a) \leq \eta N_{c_1,c_2} (x_{n-1}, x_n, a)$

$\leq \eta \max \{d_\varphi (x_{n-1}, x_n, a), c_2\varphi (x_{n-1}, Tx_{n-1}, a)/2[d_\varphi (x_{n-1}, x_n, a)+d_\varphi (x_n, x_{n+1}, a)]\}$. Consequently,

$d_\phi (x_n, x_{n+1}, a) \leq \eta d_\varphi (x_{n-1}, x_n, a)$ or $d_\phi (x_n, x_{n+1}, a) \leq \eta c_2\varphi (x_{n-1}, x_{n+1}, a) d_\varphi (x_{n-1}, x_n, a)/2 - \eta c_2\varphi (x_{n-1}, x_{n+1}, a)$,

for every n ε N. Thus

$d_\phi (x_n, x_{n+1}, a) \leq \max \{\eta, \eta c_2\varphi (x_{n-1}, x_{n+1}, a)/2 - \eta c_2\varphi (x_{n-1}, x_{n+1}, a)\} d_\varphi (x_{n-1}, x_n, a)$,

$\leq \eta d_\varphi (x_{n-1}, x_n, a)$.

Thus , the sequence $\{x_n\}_{n\varepsilon N}$ satisfies equation 3.13. Hence from lemma 3.5, we conclude that $\{x_n\}_{n\varepsilon N}$ is Cauchy sequence.

Theorem 3.10 Let (X, d_φ) be a complete extended b_2 – metric space. LetT : X→ C B(X) be a multi- valued mapping having the property that there exists $c_1, c_2 \varepsilon [0,1]$ and $\eta \varepsilon$ [0,1] such that

(i) For each $x_0 \varepsilon$ X, $\lim_{n,m\to\infty} \eta c_2 \phi (x_n, x_m, a) < 1$, here $x_n = T^n x_0$,

(ii) $H_\phi (Tx, Ty, a) \leq \eta N_{c_1,c_2} (x, y, a)$ for all x, y,aε X.(iii) T is continuous.

Then T has a fixed point in X.

Proof-From Theorem 3.11, by taking in account condition (i) and (ii), we conclude that $\{x_n\}_{n\varepsilon N}$ is a Cauchy sequence such that

$X_{n+1} = Tx_n$, …………………… 3.16

for everynεN . As X is complete, so there exists x ε x such that $\lim x_n = x$.

From inequality 3.16, by the continuity of T, it follows that

$X_{n+1} = Tx_n \to Tx$ as $n\to\infty$.

Therefore, x ε Tx, Hence T has a fixed point in X.

Theorem 3.11 Let (X, d_φ) be a complete extended b_2 – metric space. LetT : X→ C B(X) be a multi- valued mapping having the property that there exists $c_1, c_2 \varepsilon [0,1]$ and $\eta \varepsilon$ [0,1] such that

(i) For each $x_0 \varepsilon$ X, $\lim_{n,m\to\infty} \eta c_2 \phi (x_n, x_m, a) < 1$, here $x_n = T^n x_0$,

(ii) $H_\phi(Tx, Ty, a) \leq \eta N_{c_1,c_2} (x, y, a)$, for all x, y,aε X. (iii) T is x - continuous.

Then T has a fixed point in X.

Proof-From Theorem 3.11, by taking in account condition (i) and (ii), we conclude that $\{x_n\}_{n\varepsilon N}$ is a Cauchy sequence such that

$X_{n+1} = Tx_n$, …………………… 3.17

for everynεN . As X is complete, so there exists x ε x such that $\lim x_n = x$. Then we have

$d_\phi(x_{n+1}, Tx, a) = d_\phi(Tx_n, Tx, a) \leq H_\phi(Tx_n, Tx, a)$

$\leq \eta N_{c_1,c_2}(x_n, x, a) \leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, Tx_n, a), c_1 d_\varphi(x, Tx, a), c_2/2[d_\varphi(x_n, Tx, a) + d_\varphi(x, Tx_n, a)\}$

$\leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, x_{n+1}, a), c_1 d_\varphi(x, Tx, a), c_2/2[d_\varphi(x_n, Tx, a) + d_\varphi(x, Tx_n, a)\}$.

$\leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, x_{n+1}, a), c_1 d_\varphi(x, Tx, a), c_2/2\, \varphi(x_n, Tx, a)[d_\varphi(x_n, Tx, a) + d_\varphi(x_n, x, a) +$
$d_\varphi(x, Tx, a) + d_\varphi(x, x_{n+1}, a)]\}$

$\leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, x_{n+1}, a), c_1 d_\varphi(x, Tx, a), c_2/2\, \varphi(x_n, Tx, a)[d_\varphi(x_n, x, a) + d_\varphi(x, x_{n+1}, a) +$
$d_\varphi(x, Tx, a)]\}$, ………………………………….. 3.18

for every $n \varepsilon N$. Since $\lim x_n = x$ and $\lim_{n\to\infty} d_\varphi(x_n, x_{n+1}, a) = 0$. Then
$\lim_{n\to\infty} d_\varphi(x_{n+1}, Tx, a) = d_\varphi(x, Tx, a)$.
Therefore, by taking limit $n \to\infty$ in equation 3.15, we have
$d_\varphi(x, Tx, a) \leq \eta N_{c_1,c_2}(x_n, x, a)$
$\leq \eta \max\{0, , c_1 d_\varphi(x, Tx, a), c_2 \lim_{n\to\infty} \varphi(x_n, Tx, a) d_\varphi(x, Tx, a)/2\}$
$\leq \eta \max\{\eta c_1, \eta c_2 \lim_{n\to\infty} \varphi(x_n, Tx, a)/2\}\, d_\varphi(x, Tx, a)$.
As $\max\{\eta c_1, \eta c_2 \lim_{n\to\infty} \varphi(x_n, Tx, a)/2\} < 1$, so from above inequality
$d_\varphi(x, Tx, a) < d_\varphi(x, Tx, a)$, which is a contradiction, therefore $d_\varphi(x, Tx, a) = 0$ ie $x \varepsilon Tx$. Hence T has a fixed point in X.

Theorem 3.12 A multi- valued mapping $T : X \to CB(X)$ has a fixed point in a complete extended b_2 – metric space(X, d_φ), if it satisfies the following two axioms,

(i) There exists $c_1, c_2 \varepsilon [0,1]$ and $\eta \varepsilon [0,1]$ such that

$H_\phi(Tx, Ty, a) \leq \eta N_{c_1,c_2}(x, y, a)$, for all $x, y, a \varepsilon X$. ………………. 3.19

(ii) For each $x_0 \varepsilon X$, $\max\{\eta c_1 \lim_{n,m\to\infty} \phi(x_n, x_m, a), \eta c_2 \lim_{n,m\to\infty} \phi(x_n, x_m, a)\} < 1$, here $x_n = T^n x_0$…… 3.20

Proof- From Theorem 3.10, by taking in account condition (i) and (ii), we conclude that $\{x_n\}_{n\varepsilon N}$ is a Cauchy sequence such that

$X_{n+1} = Tx_n,$ …………………… 3.21

for every $n \varepsilon N$. As X is complete, so there exists $x \varepsilon x$ such that $\lim x_n = x$. Then for every $n \varepsilon N$, we have

$d_\phi(x_{n+1}, Tx, a) = d_\phi(Tx_n, Tx, a) \leq H_\phi(Tx_n, Tx, a)$

$\leq \eta N_{c_1,c_2}(x_n, x, a) \leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, Tx_n, a), c_1 d_\varphi(x, Tx, a), c_2/2[d_\varphi(x_n, Tx, a) + d_\varphi(x, Tx_n, a)\}$

$\leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, x_{n+1}, a), c_1 d_\varphi(x, Tx, a), c_2/2[d_\varphi(x_n, Tx, a) + d_\varphi(x, Tx_n, a)\}$.

$\leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, x_{n+1}, a), c_1 d_\varphi(x, Tx, a), c_2/2\, \varphi(x_n, Tx, a)[d_\varphi(x_n, Tx, a) + d_\varphi(x_n, x, a) +$
$d_\varphi(x, x_{n+1}, a) + d_\varphi(x, Tx, a)]\}$

$\leq \eta \max\{d_\varphi(x_n, x, a), c_1 d_\varphi(x_n, x_{n+1}, a), c_1 d_\varphi(x, Tx, a), c_2/2\, \varphi(x_n, Tx, a)[d_\varphi(x_n, x, a) + d_\varphi(x, x_{n+1}, a) +$
$d_\varphi(x, Tx, a)]\}$. ………………………………….. 3.22

Now, we will take two cases:

Case(I) If d_φ (x, T x, a)≤ $\lim_{n\to\infty}\sup d_\phi$ (x_n, Tx, a), then there exists a subsequence $\{x_{n_l}\}_{n\varepsilon N}$ of $\{x_n\}$ such that

d_φ (x, T x, a)≤ $\lim_{l\to\infty} d_\phi$ ($x_{n_{l+1}}$, Tx, a), so for each ε> 0, there exists l_εε N such that for every l ε N, l ≥l_ε, we have, d_φ (x, T x, a) –ε≤ d_ϕ ($x_{n_{l+1}}$, Tx, a) ≤ηmax{ d_ϕ (x_{n_l}, x, a), c_1 d_ϕ (x_{n_l},$x_{n_{l+1}}$, a),$c_1 d_\phi$ (x, Tx, a), c_2 {dϕ (x_{n_l}, Tx, a)+ d_ϕ ($x_{n_{l+1}}$, x, a) /2}} ≤ηmax{ d_ϕ (x_{n_l}, x, a), c_1 d_ϕ (x_{n_l},$x_{n_{l+1}}$, a),$c_1 d_\phi$ (x, Tx, a), c_2ϕ (x_{n_l}, Tx, a){ dϕ (x_{n_l}, x, a)+ d_ϕ ($x_{n_{l+1}}$, x, a) +d_ϕ (x, Tx, a) /2}}. ……………………… 3.23 Since $\lim_{n\to\infty} x_{n_l}$= x,$\lim_{l\to\infty} d_\phi$ (x_{n_l},$x_{n_{l+1}}$, a) = 0. Therefore, by taking lim n→∞ in equation 3.23, we haved_φ (x, T x, a) –ε≤ η max{0, $c_1 d_\phi$ (x, Tx, a), $c_2 \lim_{l\to\infty}$ϕ (x_{n_l}, Tx, a) d_ϕ (x, Tx, a)/2}

≤ η max{ c_1 , $c_2 \lim_{l\to\infty}$ϕ (x_{n_l}, Tx, a)/2 }d_ϕ (x, Tx, a)For every ε> 0. Thus d_φ (x, T x, a)≤ max{η c_1 ,η$c_2 \lim_{l\to\infty}$ϕ (x_{n_l}, Tx, a)/2 }d_ϕ (x, Tx, a).As max{η c_1 ,η$c_2 \lim_{l\to\infty}$ϕ (x_{n_l}, Tx, a)/2 } < 1, so from abov inequalityd_φ (x, T x, a)≤d_φ (x, T x, a), which is impossible,therefored_φ (x, T x, a) = 0 ie x ε Tx. Hence T has a fixed point in X.

Case(II) If d_φ (x, T x, a)> $\lim_{n\to\infty}\sup d_\phi$ (x_n, Tx, a), then there exists N_0 ε N such that for every n ≥ N_0, we have dϕ(x_{n_l}, Tx, a) ≤ d_φ (x, T x, a).From the triangle inequality

d_φ (x, T x, a)≤ϕ (x, Tx, a)[dϕ (x, Tx, x_{n+1}) + dϕ (x, x_{n+1},a) + dϕ (x_{n+1},Tx, a)] by lemma 3.5,

≤ϕ (x, Tx, a)[dϕ (x, x_{n+1},a) + dϕ (x_{n+1},Tx, a)]

We obtain

d_φ (x, T x, a) - ϕ (x, Tx, a) dϕ (x, x_{n+1},a) ≤ϕ (x, Tx, a) dϕ (x_{n+1},Tx, a)

≤ϕ (x, Tx, a) η max {d_φ (x_n, x, a),$c_1 d_\varphi$ (x_n, x_{n+1}, a),$c_1 d_\varphi$ (x, Tx, a),c_2/2[d_φ (x_n, Tx, a) + d_φ (x, Tx_n, a)}.

≤ϕ (x, Tx, a) η max {d_φ (x_n, x, a),$c_1 d_\varphi$ (x_n, x_{n+1}, a),$c_1 d_\varphi$ (x, Tx, a),c_2/2 ϕ (x, Tx, a) [d_φ (x_n, x, a) + d_φ (x, Tx, a)+d_φ (x, x_{n+1}, a) }. ………………… 3.24

Since $\lim_{n\to\infty} x_n$= x, $\lim_{n\to\infty}$ d_ϕ (x_{n+1},x_n , a) = 0. Therefore, by taking lim n→∞ in equation 3.24, we have

d_φ (x, T x, a) - ϕ (x, Tx, a) dϕ (x, x_{n+1},a) ≤ϕ (x, Tx, a) η max {0, $c_1 d_\varphi$ (x, T x, a),c_2/2$\lim_{n\to\infty}$ϕ (x_n, Tx, a) d_φ (x, Tx, a)}

≤ ϕ (x, Tx, a) max {η c_1 ,η $c_2 \lim_{n\to\infty}$ φ (x_n, Tx, a) /2} d_φ (x, Tx, a), ………………….. 3.25

From condition (ii), since

ϕ (x, Tx, a) max {η c_1 ,η $c_2 \lim_{n\to\infty}$ φ (x_n, Tx, a) /2} < 1,so from 3.24,

d_φ (x, Tx, a)≤d_φ (x, Tx, a), which is impossible, therefore d_φ (x, Tx, a) = 0 implies x = Tx. Hence T has a fixed point in X.

Example 3.13 Let X = { (α, 0) : α = 1/2, $1/2^2$, …, $1/2^n$, … } U { 0, 1 } CR^3 and letd_φ (x, y, z) denote the sequenceof the area of triangle with vertex x, y, z ε X. d_φ ((α, 0) ,(β, 0) ,(0, 1)) = $(\alpha - \beta)^2/4$, ϕ ((α, 0) ,(β, 0) ,(0, 1)) = α +β +1. Then X is an complete extendedb_2 - metric space. Define mapping T : X→ C B(X) as

(0, 0), α = (0, 0).

T(α, 0) =

($1/2^{n+1}$, 0) , α = $1/2^n$, n = 0, 1, 2,…

Hence T is continuous.

SinceN_{c_1,c_2} ($(1/2^n, 0)$,(0, 0), (0, 1)) = $\alpha^2/4 = 1/4\,(1/2^n)^2 = 1/4\times 2^{2n}$ for all $c_1, c_2\ \varepsilon\ [0, 1]/$ we get

$H_\varphi(T(1/2^n, 0), T(0, 0), (0, 1)) = H_\varphi((1/2^{n+1}, 0), (0, 0), (0, 1)) = 1/4\,(1/2^{n+1})^2 = 1/4\times 2^{2n+2}$ $\leq 1/4\ N_{c_1,c_2}((1/2^n, 0),(0, 0), (0, 1))$.

Where η = ½. Also for each $x_0 = (\alpha_0, 0)\ \varepsilon$ X.

$\lim_{n,m\to\infty} \eta\, c_2\, \varphi((\alpha_n, 0),(\alpha_m, 0),(0, 1)) = \lim_{n,m\to\infty} \eta\, c_2\, \varphi((1/2^n, 0),(1/2^m, 0),(0, 1)) < 1$

$= \lim_{n,m\to\infty} \eta\, c_2\, \varphi(x_n, x_m, a) < 1$.

Clearly it satisfy all the condition of theorem 3.10 and so there exists a fixed point.

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