Convex and Weakly Convex Subsets of a Pseudo Ordered Set

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Abstract: Inthispaperthenotionofconvexandweaklyconvex(w-convex)subsetsofapseudoorderedsetisintroducedandseveralcharacterizations are proved.It is proved that set of all convex subsets of a pseudo ordered set *A* forms a completelattice. Notion of isomorphism of psosets is introduced and characterization for convex isomorphic psosets is obtained.It is proved that lattice of all w-convex subsets of a pseudo ordered set *A* denoted by *WCS*(*A*) is lower semi modular.Also we have proved that for any two pseudo ordered sets *A* and A^1, w-convex homomorphism maps atoms of *WCS*(*A*)to atoms of $WCS(A^1)$. Concept of path preserving mapping is introduced in a pseudo ordered set and it is proved thateverymappingofapseudoorderedset*A*toitselfispathpreservingifandonlyif*A*isacycle.

MSC: 06B20, 06B10.

Keywords: Psoset,convexsubset,w-convexsubset,w-convexhull,lattice.

1. Introduction

Areflexiveandantisymmetricbinaryrelation Ð onaset*A*iscalleda*pseudo-order*onAand⟨*A*,Ð⟩iscalleda*pseudo-orderedset*ora*psoset*.For*a*,*b*∈*A*if*a*Ð*b*and*a*upperbound, thegreatestlowerbound(GLBormeet, denotedby∧*B*), theleastupperbound(LUBorjoin, denotedby ∨*B*)aredefinedanalogoustothecorrespondingnotionsinaposet[1].Itisshownin[3]thatanypsosetcanberegardedasadigraph(possiblyinfinite)inwhichforanypairofdistinctelements*u*and*v*thereisnodirectedlinebetween*u*and*v* or if there is a directed line from *u* to *v*, there is no directed line from *v* to *u*. Define a relation ±*B* on a subset *B* of apsoset⟨*A*,Ð⟩bysetting$b$±*B*b^1fortwoelements*b*and$b^1$of*B*ifandonlyifthereisadirectedpathin*B*from*b*tob^1say$b=b_0$ Ðb_1Ð··· Ð$b_n=b^1$forsome $n\geq 0$.Therelation±*B*isdefineddually. Ifforeachpairofelements*b*and$b^1$of*B*atleastoneoftherelations$b$±*B*b^1orb^1±*B*$b$holds,then*B* willbecalleda*pseudochain*ora*p-chain*.Ifforeachpair ofelements*b*and$b^1$of*B*boththerelations$b$±*B*b^1andb^1±*B*$b$hold,then*B* will be called a *cycle*. The empty set and a single element set in a psoset are cycles. A non-trivial cycle contains at least threeelements.Apsosetissaidtobe*acyclic*ifitdoesnotcontainanynon-trivialcycle.

2. ConvexSubsetsofaPsoset

Definition2.1. *AsubsetSofapseudoorderedsetAissaidtobeaconvexsubsetofAwhenevera,b*∈*Sandc*∈*Assuchthata*Ð*c*Ð*bthenc*∈*S.* Setofallconvexsubsetsofapsoset*A*isdenotedby*CS*(*A*).Clearlytheemptysetφandthepsoset*A*areconvexsubsets.Anysingletonisaconvexsubsetof*A*.Obviously⟨*CS*(*A*),⊆⟩isaposetwhere⊆isthesetinclusionrelationdefinedon*S*.For$K_1,K_2\in CS(A)$,define$K_1\wedge K_2=K_1\cap K_2andK_1\vee K_2$=smallestconvexsubsetof*A*containing$K_1\cup K_2$.Then

$\langle CS(A), \subseteq\rangle$is a complete lattice with smallest element φ and greatest element A. The lattice $\langle CS(A), \subseteq\rangle$is atomistic. Oneelementsubsetsof$A$areatomsof$CS(A)$andeachelementof$CS(A)$differentfrom φisajoinofsomeatoms.

Example 2.2. *Consider the psoset* $\langle A,Đ\rangle$*represented in Figure 1.*
$CS(A)=\{\varphi,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{b,c,d\},A\}$ *and*

$\langle CS(A),\subseteq\rangle$*isalatticewhichisrepresentedinFigure2Where*$P=\varphi,Q=\{a\},R=\{b\},S=\{c\},T=\{d\},U=\{a,b\},V=$

$\{b,c,d\}$*and*$W=A$.

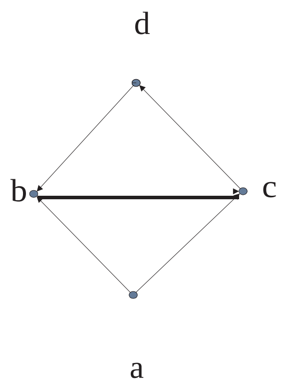

Figure1:

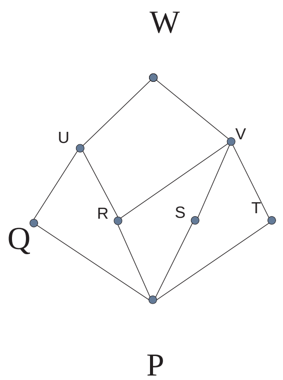

Figure2:

Remark2.3.$\langle CS(A),\subseteq\rangle$*ofFigure2isanonmodularlattice.But*$CS(A)$*isbothlowersemimodularand upper semimodular.*

Definition 2.4. *Let* $\{X_i : i \in I\}$ *be an arbitrary collection of subsets of a pseudo ordered set A. The set of all* $Z \in CS(A)$*such that* $X_i \subset Z$ *for each* $i \in I$ *will be denoted by* $CS_A(X_i : i \in I)$. *If* $\{X_i : i \in I\} = \{X, Y\}$, *we denote* $CS_A(X_i : i \in I) = CS_A(X,Y)$*and for*$X=\{a\}$*and*$Y=\{b\}$,*wedenoteitby*$CS_A(a,b)$.

Definition2.5.*LetAbeanypsosetandM*$\subseteq$*A.* *Let*$CS_A(M)=\cap\{K_i:i\in I\}$*where*$K_i$ *runsoverallconvexsubsetsof*

*AcontainingM.*Wewrite$CS_A(a,b)$for$CS_A(\{a,b\})$.

Definition 2.6.*We say that two psosets A and* A^1*are convex isomorphic if and only if* $\langle CS(A), \subseteq\rangle$*and* $\langle CS(A^1), \subseteq\rangle$*areisomorphic*.Let F be a mapping from A into B and $\varphi \neq C \subseteq A$.Denote by F/C, the restriction of F onto the subset C. That is $F/C = F\cap(C\times B)$.Alloneelementsubsetsof$A$areatomsinthelattice$\langle CS(A),\subseteq\rangle$.If$F$isanisomorphismbetween$\langle CS(A),\subseteq\rangle$and $\langle CS(A^1), \subseteq\rangle$, then as every isomorphism of atomic lattices maps atoms onto atoms, we get $F(\{a\}) = \{a^1\} \in CS(A^1)$where$a^1\in A^1$.

Definition2.7.*LetFbeanisomorphismoflattices*$\langle CS(A),\subseteq\rangle$*and*$\langle CS(A^1),\subseteq\rangle$*.Letfbeamappingfrom AtoA*^{1}*suchthat*$\{f(a)\}=F(\{a\})$*foreacha*$\in$*A.Thenwesaythatthemappingfisassociatedwiththeisomorphism*F.

Denote $f(S)=\{f(x)|x\in S\}$ for a subset S of A.

Lemma 2.8. *$F(S)=f(S)$ for any $S\in CS(A)$.*

Proof. If $a\in S$ then $\{a\}\subseteq S$ and $F(\{a\})=\{f(a)\}\subseteq F(S)$ as F is an isomorphism. Then $f(a)\in F(S)$ and thus $f(S)\subseteq F(S)$. Conversely, if $a^1\in F(S)$ then $\{a^1\}\subseteq F(S)$ and $F^{-1}(\{a^1\})=\{f^{-1}(a^1)\}\subseteq S$ as F^{-1} is also an isomorphism. Then $f^{-1}(a^1)\in S$ and $a^1\in f(S)$ so that $F(S)\subseteq f(S)$ proving that $F(S)=f(S)$. □

Theorem 2.9. *Let f be associated with an isomorphism F of the lattices $\langle CS(A),\subseteq\rangle$ and $\langle CS(A^1),\subseteq\rangle$. Then*

$f(CSA(M))=CS_{A}1(f(M))$ for any subset $M\subseteq A$.

Proof. As $M\subseteq \bigcap CSA(M)$, we have $f(M)\subseteq f(CSA(M))$. Now, by lemma 2.8 $f(\bigcap CSA(M)) = F(\bigcap CSA(M))\in CS(A^1)$ and therefore $CS_{A}1(f(M))\subseteq f(CS_A(M))$. On the other hand, let $Z\in CS(A^1)$ be such that $f(M)\subseteq Z$. Since F is surjective, there exists $W\in CS(A)$ with $F(W)=f(W)=Z$. It follows that $M\subseteq W$ and therefore $\bigcap CSA(M)\subseteq W$, consequently $f(\bigcap CS_A(M))\subseteq Z$ and $f(CS_A(M))\subseteq CS_{A}1(f(M))$. □

Theorem 2.10. *The following three conditions are equivalent for two psosets A and A^1.*
(i). The psosets A and A^1 are convex isomorphic.
(ii). There exists a bijection $f:A\longrightarrow A^1$ such that $f(CS_A(M))=CS_{A}1(f(M))$ for $M\subseteq A$.
(iii). There exists a bijection $f:A\longrightarrow A^1$ such that $f(CS_A((a,b)))=CS_{A}1((f(a),f(b)))$ for each $a,b\in A$.

Proof. $(i)\Rightarrow(ii)$: follows from Theorem 2.9.

(ii) $\Rightarrow(iii)$: Follows directly.

(iii) $\Rightarrow(i)$: Let f be a bijection satisfying (iii). Denote by $P(A)$ the power set of A and define a mapping $F : P(A) \to P(A^1)$ such that $F(S)=f(S)$ for each $S\in P(A)$. Now, we prove that for any convex set S, its image $F(S)$ is also convex. Clearly $f(a), f(b)\in f(S) = F(S)$ for each $a, b\in S$. If $S\in CS(A)$ then $CSA(a, b)\subseteq S$ for arbitrary $a, b\in S$ and so by (iii), we have $CS_{A}1(f(a), f(b)) = f(CS_A(a, b))\subseteq f(S) = F(S)$. This implies that the mapping F maps convex subsets of A onto convex subsets of A^1 and F is a bijection as f is a bijection. Therefore the restriction of the mapping $F/CS(A): CS(A)\longrightarrow CS(A^1)$ is also a bijection. Since $S\subseteq T$ if and only if $F(S)\subseteq F(T)$ for each $S,T\in CS(A)$, the mapping $F/CS(A)$ is an isomorphism of lattices $\langle CS(A),\subseteq\rangle$ and $\langle CS(A^1),\subseteq\rangle$. Therefore psosets A and A^1 are convex isomorphic. □

3. w-convex Subsets of a Psoset

Definition 3.1. *A subset S of a psoset A is said to be a w-convex subset (weakly convex subset) of A whenever $a,b\in S$ and $c\in A$ such that $a\pm_A c\pm_A b$ then $c\in S$.*

Setofallw-convexsubsetsofapsoset$A$isdenotedby$WCS(A)$anditformsalatticewithrespecttotherelation$\subseteq$.

Remark3.2.

(1). For$H_1,H_2\in WCS(A)$,define$H_1\wedge H_2=H_1\cap H_2andH_1\vee H_2$=thesmallestw-convexsubsetof$A$containing$H_1\cup H_2$.(2).$\langle WCS(A),\subseteq\rangle$isacompletelatticeas$\phi$istheleastelementandAisthegreatestelementof$WCS(A)$.

Example3.3.*Apsoset$\langle A,Ð\rangle$where$A=\{a,b,c,d\}$andthelatticeofallitsw-convexsubsetsareshowninFigure3.*

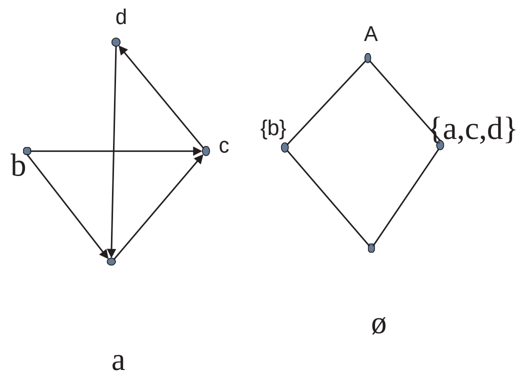

Figure3:

Definition3.4.*LetSbeasubsetofapsosetA.Thew-convexhullofSdenotedby$wch(S)$isdefinedtobethesmallestw-convexsubsetofAcontainingS.*

Theorem3.5.*LetSbeasubsetofapsosetA. Then$wch(S)=\{q\in A|p_1\pm_A q\pm_A p_2$forsome$p_1,p_2\in S\}$where$p_1,p_2$ neednot bedistinct.*

Proof. Let$Q=\{q\in A|p_1\pm_A q\pm_A p_2$forsome$p_1,p_2\in S\}$.Clearly$Q$isasubsetofanyw-convexsubsetof$A$containing S.Then$Q\subseteq wch(S)$.Letusprovethat$Q$itselfisaw-convexsubsetof$A$.Let$q_1,q_2\in Q$suchthat$q_1\pm_A r\pm_A q_2$forsome $r\in A$. Now$q_1\in Q$impliesthereexistsome$p_1,p_2\in S$suchthat$p_1\pm_A q_1\pm_A p_2$. Also$q_2\in Q$impliesthereexist$p^1_1,p^1_2\in S$ suchthat$p^1_1\pm_A q_2\pm_A p^1_2$.Then$p_1\pm_A q_1\pm_A r\pm_A q_2\pm_A p^1_2$whichimplies$r\in Q$. Therefore$Q=wch(S)$. □

Corollary3.6.*Foranyelementainacycle$C$,$wch(\{a\})=C$.*Alattice$L$issaidtobe*lowersemimodular*if$x\vee y$covers $x$and$y$imply$x$and$y$cover$x\wedge y$.

Theorem 3.7.*LatticeofallIw-convexsubsetsofapsosetAislowersemimodular.*

Proof. Let$S_1,S_2\in WCS(A)$,latticeofallw-convexsubsetsofapsoset$A$. Let$P=S_1\vee S_2$and$Q=S_1\wedge S_2$. Let$P$ coverbothS_1andS_2. ItsufficestoprovethatS_1covers$Q$. Supposethereexistsaw-convexsubset$S^1$of$A$suchthat

$Q \subseteq S^1 \subseteq S1$.Let$s0 \in S^1 - Q$.Then$s0 \in S1ands0 \notin S2$.Let$s1 \in S1 - S^1$.Then$s1 \in S1, s1 \notin S2ands1, s0$.

Now

$S2 \subset S2 \cup \{s0\} \subset P$, as P is the smallest w-convex subset of A containing $S1 \cup S2$. But P covers $S2$ imply $S2 \cup \{s0\}$ is notaw-convexsubsetof$A$.Thereforethereexistsapathbetween$s0$andanelement$k0 \in S2$whichdoesnotliecompletelyin$S2 \cup \{s0\}$.Assumethepathintheform$k0 \pm_A s0$.(similarargumentholdsifthepathisoftheform$s0 \pm_A k0$).

Let$X = \{s \in S^1 - Q | k \pm_A s$forsome $k \in S2\}$.Xisnonemptyas$s0 \in X$and$S2 \subset S2 \cup X$.Furtheras$s1 \notin X$(infact $s1 \notin S^1$),wehave$S2 \cup X \subset P$.AsPcovers$S2$,$S2 \cup X \notin WCS(A)$.Thereforethereexistsapath$m \pm_A n$betweentwo elements$m,n$of$S2 \cup X$whichisnotcontainedin$S2 \cup X$.Thisimplies$m \pm_A t \pm_A n$but$t \notin S2 \cup X$.Wecanassumethat $t \in P$as$S2 \cup X$isnotaw-convexsubsetof$P$.Inthefollowingcaseseitherwegetacontradictiontothew-convexityof$S2$orS^1itselfisnotw-convex,proving$S1$covers$Q$.

Case(1): Let$m,n \in S2$. Thisisacontradictiontothew-convexityof$S2$.

Case(2): Let$m \in X$and$n \in S2$.As$m \in X, m \notin S2$.Bythe definitionof$X$thereexistsa$k \in S2$suchthat$k \pm_A m$. Thus$k \pm_A m$and$m \pm_A t \pm_A n$imply$k \pm_A t \pm_A n$,whichcontradictsthew-convexityof$S2$.

Case(3): Let$m, n \in X$.As$m \in X$,thereexists a path $k \pm_A m$ for some $k \in S2$. But we have a path $m \pm_A t$ whichimpliesthereisapath$k \pm_A t$.But$t \notin X$,ie$t \notin S^1 - Q$.Since$t \notin S2$,itcannotbein$Q$.So$t \notin S^1$.But$m,n \in X \subseteq S^1$showsthatS^1isnotaw-convexsubsetof$A$.

Case(4): Let$m \in S2$and$n \in X$.Aswehaveapathfrom$m \pm_A t$, $t \notin X = S^1 - Q$andsince$t \notin S2$imply$t \notin S^1$.Now$P$

covers$S2$and$t \notin S2$,wemusthave$wch(S2 \cup \{t\}) = P$.Thuseveryelementof$P$isin$S2$orelseliesonsomepathbetween $t$andanelementof$S2$.Inparticularconsidersome$s0 \in X$,wehave$k0 \pm_A s0$and$t \pm_A n$.If$k \pm_A s0 \pm_A t$where$k \in S2$,then we have $s0 \pm_A t \pm_A n$ which proves that S^1 is not w-convex.On the other hand if $t \pm_A s0 \pm_A k$ then $k0 \pm_A s0 \pm_A k$,contradictingthew-convexityof$S2$. □

Theorem3.8.*IfScoversS*^{1}*inWCS*(A)*andp,qbelongtoS*$-S^1$*thenwch*$(\{p\}) = wch(\{q\})$.

Proof.
AsScoversS^1,$wch(S^1 \cup \{p\}) = wch(S^1 \cup \{q\}) = S$.Then$p$liesinapathfrom$qtor$where$r \in S^1andq$liesinapathfrom$ptos$where$s \in S^1$.Ifthereexistpaths$p \pm_A qandq \pm_A p$thentheproofisdone.Butifbothpaths havethesamedirectionsay$p \pm_A q$,wehavepaths$p \pm_A r$and$s \pm_A p$with$r, s \in S^1$,contradictingthew-convexityofS^1. □

Definition 3.9. *Let* $\langle A, Đ\rangle$*and* A^1, Đ^1*be any two psosets. A mapping* f : $A \longrightarrow A^1$*is called(1).orderpreservingiffora,b*$\in A$*,a*Đ*bimpliesf*(a)Đ$^1 f(b)$.
(2). pathpreservingiffora,b$\in A$*,a*$\pm_A$*bimpliesf*$(a) \pm_{A^1} f(b)$.

Remark 3.10.*Any order preserving mapping f is path preserving. The converse is not true. For example, in the psosetA of Figure 4, define a mapping f: A* $\longrightarrow$ *A by f* $(a) = b$, *f* $(b) = a$, *f* $(c) = c$.*Clearly f is path preserving but not orderpreserving.*

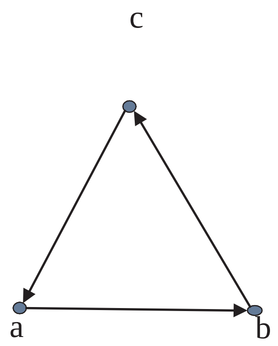

Figure4:

Theorem3.11.*EverymappingofapsosetAtoitselfispathpreservingifandonlyifAisacycle.*

Proof. IfAisacycle,thenforanytwoelements$a,b$of$A$,both$a\pm_A b$and$b\pm_A a$hold. Thereforeeverymappingof$A$ toitselfispathpreserving.Conversely,letusassumethat$A$isnotacycle.Thenthereexistsatleastonepairofeleme nts say(a,b)in$A$suchthat$a\pm_A b$holdsbut$b\pm_A a$doesnothold.
Define$f$:$A\longrightarrow A$by$f(b)=a$and$f(c)=b$forall$c$ □
b
.Then$f$isnotpathpreservingas$a\pm_A b$but$f(a)\pm_A f(b)$doesnothold.

Onecaneasilyprovethefollowingtheorem.

Theorem3.12.*Let$f$:$A\longrightarrow A^1$bepathpreserving. IfSisaw-convexsubsetinAthen$f(S)$isaw-convexsubsetinA^1.*

Definition3.13.*Let$\langle A,Ð\rangle$andA^1,$Ð^1$beanytwopsosets.Amapping$f$:$A\to A^1$iscalledahomomor phismif(1).fisorderpreserving.*
(2).$a^1Ð^1b^1$inA^1impliesthereexists$a\in f^{-1}(a^1)$and$b\in f^{-1}(b^1)$suchthat$aÐb$.

Theorem3.14.*Let$f$:$A\to A^1$beahomomorphism.IfS^1isaw-convexsubsetofA^1then$f^{-1}(S^1)$isaw-convexsubsetof$A$.*

*Proof.*Let$a,b\in f^{-1}(S^1)$suchthat$a\pm_A c\pm_A b$forsome$c\in A$.If$c\notin f^{-1}(S^1)$then$f(c)\notin S^1$.Now$f(a)\pm_{A^1} f(c)\pm_{A^1} f(b)andf(c)\notin S^1$,acontradictiontothew-convexityofS^1.Hence$c\in f^{-1}(S^1)$and$f^{-1}(S^1)$isw-convex. □

Remark3.15.*Iff:$A\longrightarrow A^1$isahomomorphismbetweentwopsosetsAandA^1andifS isaw-convexsubsetofAthen$f(S)$neednotbeaw-convexsubsetofA^1.For,inFigure5defineamap$f$:$A\longrightarrow A^1$by$f(a)=w,f(b)=y,f(c)=x,f(d)=z$.Clearlyfis ahomomorphism.Observethat$\{b\}$isw-convexinAwhereas$f(\{b\})=\{y\}$isnotw-convexinA^1.*

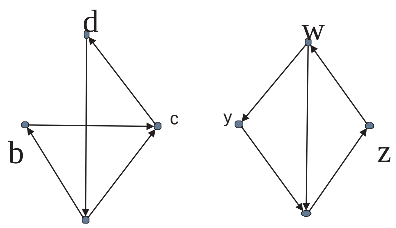

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Figure5:

Definition3.16.*AhomomorphismbetweentwopsosetsAandA*^{1}*iscalledaw-convexhomomorphismifittakesw-convexsubsetsofAontow-convexsubsetsofA*^{1}.

Anelement$a$ofalattice$L$issaidtobean*atom*ifforany$b\in L$,$0\leq b\leq a$implyeither$b$=0or$b$=$a$where0istheleast elementof$L$.

Remark3.17.

(1). *AcycleinapsosetAisalwaysanatomof$WCS(A)$*

(2). *w-convexhullofasingleelementinapsosetAisanatomin$WCS(A)$.*

Theorem 3.18. *Let $f:A\rightarrow A^1$be a w-convex homomorphism. If S is an atom in $WCS(A)$ then $f(S)$ is an atom in*

$WCS(A^1)$.ConverselyifS^{1}isanatomin$WCS(A^1)$thenthereexistsanatomSin$WCS(A)$suchthat$f(S)=S^1$.

Proof. If$f(S)$isnotanatomin$WCS(A^1)$thenthereexistsaw-convexsubsetS^1in$WCS(A^1)$suchthat$\phi\subset S^1\subset f(S)$.But then $\phi\subset f^{-1}(S^1)\cap S\subset S$, where $f^{-1}(S^1)\cap S$ is also a w-convex subset of A, contradicting the fact that S is an atom.Conversely,letS^1beanatomin$WCS(A^1)$and$S\subseteq f^{-1}(S^1)$beanatomin$WCS(A)$.Then$\phi\subseteq f(S)\subseteq S^1$andsince$S^1$isanatomin$WCS(A^1)$,wehave$f(S)=S^1$. □

Corollary3.19.*Anyw-convexhomomorphismmapsacyclicpsosetsintoacyclicpsosets.*

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