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Some Aspects Of 2-Fuzzy 2-Inner Product Spaces

Thangaraj Beaula[1], R.Abirami[2]

[1]PG and Research Department of Mathematics,T.B.M.L.College, Porayar-609307
(Affiliated to Bharathidasan University, Tiruchirappalli)
Tamilnadu, India
[1]PG and Research Department of Mathematics,D.G.G.A.College(W), Mayiladuthurai-609001
(Affiliated to Bharathidasan University, Tiruchirappalli)
Tamilnadu, India

Email: [1]edwinbeaula@yahoo.co.in, [1]mithraa.abi@gmail.com

Abstract— In this paper the concept of 2-fuzzy 2-inner product space is introduced and also the crisp norm α-2-norm corresponding to this inner product space is developed. Further parallelogram law and polarization identity in 2-fuzzy 2-inner product space are proved.

Keywords— 2-fuzzy 2-innerproduct space, α-2-norm, Polarization Identity, Parallelogram law.

1. INTRODUCTION

In 1965, Zadeh [18] introduced the idea of fuzzy sets, kicking off a new revolutionary field in mathematics. Gahler [9] presented the principle of the 2-norm on a linear space. Katsaras [10] introduced the concept of a fuzzy norm on a linear space in 1984. Chen & Mordeson [2], Bag & Samanta [1], and others have provided several definitions of fuzzy normed spaces.In defining 2-fuzzy normed linear space, Somasundaram & Thangaraj Beaula [15] established the notion of fuzzy 2- normed linear space $(F(X), N)$, and Thangaraj Beaula & Gifta[17] proved some standard results.

C.R.Diminnie, S.Gahler, and A.White [4] introduced the idea of 2-inner product space. Further definitions of fuzzy inner product space [5,11,12] and fuzzy normed linear space [6,7,8,10,13,14] were given by various writers. In [16], Vijayabalaji & Thilaigovindan proposed fuzzy n-inner product space as a generalization of the n-inner product space principle proposed by Y.J.Cho, M.Matic, and J.Pecaric in [3].

The definition of a 2-fuzzy 2-inner product space is introduced in this paper, as well as the crisp norm α-2-norm corresponding to this inner product space. In 2-fuzzy 2-inner product space, the parallelogram law and polarisation identity are also demonstrated.

2. PRELIMINARIES

Definition 2.1
A fuzzy set is defined as$A = \{x, \mu_A(x): x \in X\}$, with a membership function $\mu_A(x): X \to [0,1]$, where $\mu_A(x)$ denotes the degree of membership of the element x to the set A.

Definition 2.2

Let X be a non empty and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{(x,\mu)/x \in X \text{and } \mu \in (0,1]\}$. Clearly f is bounded function for $|f(x)| \leq 1$. Let K be the space of real numbers then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by

$f + g = \{(x,\mu) + (y,\eta)\} = \{(x+y),(\mu,\eta)/\ (x,\mu) \in f \ and\ (y,\eta) \in g\}$

and $kf = \{(kf,\mu)/\ (x,\mu) \in f\}$

where $k \in K$.

The linear space $F(X)$ is said to be normed space if for every $f \in F(X)$ there is associated a non-negative real number $\|f\|$ called the norm of f in such a way,

(1) $\|f\| = 0$ if and only if $f = 0$.

For,

$$\|f\| = 0 \Leftrightarrow \{\|(x,\mu)\|/\ (x,\mu) \in f\} = 0$$
$$\Leftrightarrow x = 0, \mu \in (0,1] \Leftrightarrow f = 0$$

(2) $\|kf\| = |k|\|f\|,\ k \in K$.

For ,

$$\|kf\| = \{\|k(x,\mu)\|/\ (x,\mu)f, k \in K\}$$
$$= \{|k|\|x,\mu\|/\ (x,\mu) \in f\} = |k|\|f\|$$

(3) $\|f + g\| < \|f\| + \|g\|$ for every $f, g \in F(X)$.

For,

$$\|f + g\| = \{\|(x,\mu) + (y,\eta)\|/\ \mathrm{x, y} \in \mathrm{X}, \mu, \eta \in (0,1]\}$$
$$= \{\|(x+y),(\mu \wedge \eta)\|/\ \mathrm{x, y} \in \mathrm{X}, \mu, \eta \in (0,1]\}$$
$$\leq \{\|(x, \mu \wedge \eta)\| + \|(y, \mu \wedge \eta)\|/\ (\mathrm{x}, \mu) \in \mathrm{f}\ \text{and} (\mathrm{y}, \eta) \in \mathrm{g}\}$$

$= \|f\| + \|g\|$

Then $(F(X), \|.\|)$ is a normed linear space.

Definition 2.3

A 2-fuzzy set on X is a fuzzy set on $F(X)$.

Definition 2.4

Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R(R$, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,

(N1) for all $t \in R$ with $t \leq 0$, $N(f_1, f_2, t) = 0$.

(N2) for all $t \in R$ with $t \geq 0$, $N(f_1, f_2, t) = 1$ if and only if f_1 and f_2 are linearly dependent.

(N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(N4)for all $t \in R$, with $t \geq 0$, $N(f_1, cf_2, t) = N(f_1, f_2, t/|c|)$ if $c \neq 0, c \in K$ (field).

(N5) for all $s, t \in R$, $N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$.

(N6) $N(f_1, f_2, .)$: $(0,\infty) \rightarrow [0,1]$is continuous.

(N7) $\lim_{t\to\infty} N(f_1, f_2, t) = 1$.

Then (F(X),N) is a fuzzy 2-normed linear space or (X,N) is a 2-fuzzy 2-normed linear space.

Definition 2.5

A sequence $\{f_n\}$ in a 2-fuzzy normed linear space (F(X),N) is said to be a convergent sequence if for a given $t > 0$ and $0<r<1$ there exist a positive number $n_0 \in N$ such that

$N(f_n - f, g, t) > 1 - r$ for $g \in F(X)$ and for every $n \geq n_0$.

Definition2.6

A sequence $\{f_n\}$ is said to be a Cauchy sequence in a 2-fuzzy normed linear space $F(X)$ if for agiven $r > 0$ with $0 < r < 1$, $t > 0$ there exist a positive number n_0 such that

$N(f_n - f_m, g, t) > 1 - r$ for $g \in F(X)$ and for every n, m $\geq n_0$.

Definition 2.7
A 2-fuzzy 2-normed linear space (X,N) is said to be complete if every Cauchy sequence in X converge to some point in X.

3. 2-Fuzzy 2-Inner Product Space

Definition 3.1
Let $F(X)$ be a linear space over the complex field $\mathbb{C}$. Define a fuzzy subset μ as a mapping from $F(X) \times F(X) \times F(X) \times \mathbb{C} \to [0,1]$ such that $f_1 \in F(X)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying the following conditions
(I_1) For $f, g, h \in F(X)$ and $s, t \in \mathbb{C}$
$\mu(f+g, h, f_1, |t|+|s|) \geq \min\{\mu(f, h, f_1, |t|), \mu(g, h, f_1, |s|)\}$.
(I_2) For $s, t \in \mathbb{C}$, $\mu(f, g, h, |st|) \geq$
$\min\{\mu(f, f, h, |s|^2), \mu(g, g, h, |t|^2)\}$.
(I_3)For $t \in \mathbb{C}, \mu(f, g, h, |t|) = \mu(g, f, h, |t|)$.
(I_4) For $\alpha_1, \alpha_2 \in \mathbb{C}$with $\alpha_1 \neq 0, \alpha_2 \neq 0$,
$\mu(\alpha_1 f, \alpha_2 g, h, t) = \mu\left(f, g, h, \frac{t}{|\alpha_1 \alpha_2|}\right)$.
(I_5) $\mu(f, f, h, t) = 0 \; \forall \; t \in \mathbb{C}/R^+$
$\mu(f, f, h, t) = 1 \; \forall \; t > 0$if and only if f, h are linearly dependent.
$(I_6)\mu(f, g, h, t)$ is invariant under any permutation.

$$(I_7)\forall \; t > 0, \qquad \mu(f, f, h, t) = \mu(g, g, h, t)$$

(I_8) $\mu(f, g, h, t)$ is monotonic non-decreasing function of $\mathbb{C}$ and $\lim_{t\to\infty} \mu(f, g, h, t) = 1$.
Then μ is said to be the 2-fuzzy 2-inner product on $F(X)$ and the pair $(F(X), \mu)$ is called 2-fuzzy 2-inner product space.
Example 3.2
Consider the mapping $f: S^2 \to [0,1]$ where S^2 is an 2-dimensional unit sphere defined as $f(x_1, x_2) = |1 - (x_1^2 + x_2^2)|$ define an 2-dimensional inner product as
$\langle f, g, h \rangle = \det(A)$, where $A = \begin{bmatrix} f.g & f.h \\ h.g & h.h \end{bmatrix}$
Where $f.g$ represents the usual inner product between two functions with

$$f.g = \int f(x)g(x)dx,$$

where $x = (x_1, x_2)$
Then $(F(X), \langle .,. \rangle)$ is an 2-inner product space.
By considering

$$\mu(f, g, h, t) = \begin{cases} \dfrac{t}{t + \langle f, g, h, t \rangle} & when\; t > 0 \\ 0 & when\; t \in \mathbb{C}/R^+ \end{cases}$$

The space $(F(X), \mu)$ is a 2-fuzzy 2-inner product space
Proof :
(I_1) For $f, g, h \in F(X)$ and $s, t \in \mathbb{C}$

$$\mu(f+g, h, f_1, |t|+|s|) \geq \min\{\mu(f, h, f_1, |t|), \mu(g, h, f_1, |s|)\}$$

If
(a) $s + t < 0$,
(b) $s = t = 0; s > 0, t < 0 \; (or) \; s < 0, t > 0$,
(c) $s + t > 0; s, t \geq 0$
The proof is obvious.

Consider
(d) $s > 0, t > 0, s + t > 0$
Without loss of generality assume

$$\mu(f, h, f_1, |t|) \geq \mu(g, h, f_1, |s|)$$
$$\Rightarrow \frac{t}{t + \langle f, h, f_1 \rangle} \leq \frac{s}{s + \langle g, h, f_1 \rangle}$$
$$\Rightarrow \frac{t + \langle f, h, f_1 \rangle}{t} \geq \frac{s + \langle g, h, f_1 \rangle}{s}$$
$$\Rightarrow 1 + \frac{\langle f, h, f_1 \rangle}{t} \geq 1 + \frac{\langle g, h, f_1 \rangle}{s}$$
$$\Rightarrow \frac{\langle f, h, f_1 \rangle}{t} \geq \frac{\langle g, h, f_1 \rangle}{s}$$
$$\Rightarrow \frac{s \langle f, h, f_1 \rangle}{t} \geq \langle g, h, f_1 \rangle$$
$$\Rightarrow \langle f, h, f_1 \rangle + \frac{s\langle f, h, f_1 \rangle}{t} \geq \langle f, h, f_1 \rangle + \langle g, h, f_1 \rangle$$
$$\Rightarrow \left(1 + \frac{s}{t}\right) \langle f, h, f_1 \rangle \geq \langle f, h, f_1 \rangle + \langle g, h, f_1 \rangle$$
$$\Rightarrow \left(\frac{s + t}{t}\right) \langle f, h, f_1 \rangle \geq \langle f, g, h, f_1 \rangle$$
$$\Rightarrow \frac{\langle f, h, f_1 \rangle}{t} \geq \frac{\langle f, g, h, f_1 \rangle}{s + t}$$
$$\Rightarrow 1 + \frac{\langle f, h, f_1 \rangle}{t} \geq 1 + \frac{\langle f, g, h, f_1 \rangle}{s + t}$$
$$\Rightarrow \frac{t + \langle f, h, f_1 \rangle}{t} \geq \frac{s + t + \langle f, g, h, f_1 \rangle}{s + t}$$

$\Rightarrow \frac{t}{t+\langle f,h,f_1 \rangle} \leq \frac{s+t}{s+t+\langle f,g,h,f_1 \rangle}$
$\Rightarrow \mu(f + g, h, f_1, |t| + |s|) \geq \min\{\mu(f, h, f_1, |t|), \mu(g, h, f_1, |s|)\}$
(I_2) For $s, t \in \mathbb{C}$,
$\mu(f, g, h, |st|) \geq \min\{\mu(f, f, h, |s|^2), \mu(g, g, h, |t|^2)\}$
Without loss of generality $\mu(f, f, h, |t|), \mu(g, g, h, |s|)$

$$\Rightarrow \frac{t}{t + \langle f, f, h \rangle} \leq \frac{s}{s + \langle g, g, f_1, h \rangle}$$
$$\Rightarrow \frac{t + \langle f, f, h \rangle}{t} \geq \frac{s + \langle g, g, f_1, h \rangle}{s}$$
$$\Rightarrow 1 + \frac{\langle f, f, h \rangle}{t} \geq 1 + \frac{\langle g, g, f_1, h \rangle}{s}$$
$$\Rightarrow \frac{\langle f, f, h \rangle}{t} \geq \frac{\langle g, g, f_1, h \rangle}{s}$$
$$\Rightarrow \frac{s \langle f, f, h \rangle}{t} \geq \langle g, g, f_1, h \rangle$$
$$\Rightarrow \frac{s\langle f, f, h \rangle . \langle f, f, h \rangle}{t} \geq \langle g, g, f_1, h \rangle . \langle f, f, h \rangle$$

Using,

$$|\langle f, g, h \rangle| \leq \sqrt{\langle f, f, h \rangle} . \sqrt{\langle g, g, h \rangle}$$
$$|(\langle f, f, h \rangle^2) s/t)| \geq \langle f, f, h \rangle^2$$

$$\langle f,f,h\rangle^2 s/t \geq \frac{\langle f,g,h\rangle^2}{t}$$
$$\Rightarrow \frac{\langle f,f,h\rangle^2 s}{t^2} \geq \frac{\langle f,g,h\rangle^2}{st}$$
Taking square root on both sides
$$\Rightarrow \frac{\langle f,f,h\rangle}{t} \geq \frac{\langle f,g,f_1,h\rangle}{\sqrt{st}}$$
$$\Rightarrow 1+\frac{\langle f,f,h\rangle}{t} \geq 1+\frac{\langle f,g,f_1,h\rangle}{\sqrt{st}}$$
$$\Rightarrow \frac{t+\langle f,f,h\rangle}{t} \geq \frac{\sqrt{st}+\langle f,g,f_1,h\rangle}{\sqrt{st}}$$
$$\Rightarrow \frac{t}{t+\langle f,f,h\rangle} \geq \frac{\sqrt{st}}{\sqrt{st}+\langle f,g,f_1,h\rangle}$$
$$\Rightarrow \mu(f,g,h,|st|) \geq \min\{\mu(f,f,h,|s|^2),\mu(g,g,h,|t|^2)\}$$
(I_3) For $\in \mathbb{C}$, $\mu(f,g,h,|t|)=\mu(g,f,h,|t|)$
$$\mu(f,g,h,|t|)=\frac{t}{t+\langle f,g,h\rangle}$$
$$=\frac{t}{t+\langle g,f,h\rangle}$$
$$=\mu(g,f,h,|t|)$$
(I_4) For $\alpha_1,\alpha_2 \in \mathbb{C}$ with $\alpha_1 \neq 0, \alpha_2 \neq 0$,
$$\mu(\alpha_1 f,\alpha_2 g,h,t)=\mu\left(f,g,h,\frac{t}{|\alpha_1\alpha_2|}\right)$$
$$\mu(\alpha_1 f,\alpha_2 g,h,t)=\frac{t}{t+\langle \alpha_1 f,\alpha_2 g,h\rangle}$$
$$=\frac{t}{t+|\alpha_1.\alpha_2|\langle f,g,h\rangle}$$
$$=\frac{t/|\alpha_1.\alpha_2|}{t/|\alpha_1.\alpha_2|\langle f,g,h\rangle}$$
$$=\mu\left(f,f,h,\frac{t}{|\alpha_1\alpha_2|}\right)$$
$$(I_5)\mu(f,f,h,t)=0\ \forall\ \ t\in \mathbb{C}/R^+$$
$\mu(f,f,h,t)=1\ \forall\ \ t>0$if and only if f,h are linearly dependent.
When $t\in\mathbb{C}/R^+$ by definition $\mu(f,f,h,t)=0$
When $t>0$, $\mu(f,f,h,t)=1$
$$\Leftrightarrow \frac{t}{t+\langle f,f,h\rangle}=1$$
$$\Leftrightarrow \langle f,f,h\rangle=0$$
$\Leftrightarrow f,h$ are linearly dependent.
$(I_6)\mu(f,g,h,t)$ is invariant under any permutation.
$\mu(f,g,h,t)$ is invariant under any permutation as $\langle f,g,h\rangle$ is invariant under any permutation.
$$(I_7)\forall\ \ t>0,$$
$$\mu(f,f,h,t)=\mu(g,g,h,t)$$

$$\mu(f,f,h,t) = \frac{t}{t + \langle f,f,h\rangle}$$
$$= \frac{t}{t + \langle g,g,f_1,h\rangle} = 1$$
$$= \mu(g,g,h,t)$$

(I_8) $\mu(f,g,h,t)$ is monotonic non-decreasing function of $\mathbb{C}$ and $\lim_{t\to\infty} \mu(f,g,h,t) = 1$.
If $t_1 < t_2 \le 0$,

$$\mu(f,g,h,t_1) = \mu(f,g,h,t_2) = 0$$

Assume $t_2 > t_1 > t > 0$

$$\frac{t_2}{t_2 + \langle f,g,h\rangle} - \frac{t_1}{t_1 + \langle f,g,h\rangle}$$
$$\Rightarrow \frac{\langle f,g,h\rangle\,(t_2 - t_1)}{(t_2 + \langle f,g,h\rangle)(t_1 + \langle f,g,h\rangle)} \ge 0$$

For all $\langle f,g,h\rangle \in F(X)$

$$\Rightarrow \frac{t_2}{t_2 + \langle f,g,h\rangle} \ge \frac{t_1}{t_1 + \langle f,g,h\rangle}$$
$$\Rightarrow\ \mu(f,g,h,t_2) \ge \mu(f,g,h,t_1)$$

Thus $\mu(f,g,h,t)$ is a non-decreasing function.
Also $\lim_{t\to\infty} \mu(f,g,h,t) = \lim_{t\to\infty} \frac{t}{t+\langle f,f,h\rangle}$
Therefore, $(F(X),\mu)$ is a 2-fuzzy 2-inner product space.

Definition 3.3

Let $(F(X),\mu)$ be a 2-fuzzy 2-inner product space satisfying the condition $\mu(f,g,h,t^2) > 0$ when $t > 0$ implies that $f.g$ are linearly dependent. Then all$\alpha \in (0,1)$. Define, $\|f,h\|_\alpha = \inf\{t: \mu(f,f,h,t^2) \ge \alpha\}$ a crisp norm on $F(X)$ called the α-2-norm on $F(X)$ generated by μ.

Theorem 3.4

Let μ be a α-fuzzy 2-inner product on $F(X)$. Then a fuzzy subset N defined as $N: F(X) \times R \to [0,1]$ given by

$$N(f,h,t) = \begin{cases} \mu(f,f,h,t^2) & when\ t \in R, t > 0 \\ 0 & when\ t \in R, t \le 0 \end{cases}$$

Proof:

(N_1) From definition of α-fuzzy 2-inner product space by condition (I_5) it implies that $\mu(f,f,h,t^2) = 0$ for all $t \in \mathbb{C}/R^+$ and so $N(f,h,t) = 0$ for all $t \in R, t \le 0$.
(N_2) From (I_5) for all $t > 0$, $\mu(f,f,h,t^2) = 1$ if f,h are linearly dependent therefore it follows that $N(f,h,t) = 1$ if f is linearly dependent
$(N_3)N(f,h,t)$ is invariant under any permutation of f since μ is invariant under any permutation.
(N_4) For all $t > 0$ and $c \ne 0$

$$N(cf,h,t) = \mu(cf,cf,h,t^2)$$
$$= \mu\left(f,cf,h,\frac{t^2}{|c|}\right)$$
$$= \mu\left(f,f,h,\frac{t^2}{|c|}\right)$$
$$= \mu\left(cf,h,\frac{t}{|c|}\right)$$

(N_5) To prove that
$N(f+g,h,s+t) \ge \min\{N(f,h,s),N(g,h,t)\}$for every $s,t \in R$ and $f,g \in F(X)$
Following three cases arise

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I. $s + t < 0$
II. $s = t = 0, s > 0, t < 0$ or $s < 0, t > 0$
III. $s + t > 0,\ s, t \geq 0$

To prove (iii)
Consider
$N(f + g, h, s + t) = \mu(f + g, f + g, h, (s + t)^2)$
$= \mu(f + g, f + g, h, s^2 + t^2 + 2st)$
$\geq \mu(f, f, h, s^2) \wedge \mu(g, g, h, t^2) \wedge \mu(f, g, h, st)$
$\geq \mu(f, f, h, s^2) \wedge \mu(g, g, h, t^2)$
$= N(f, h, s) + N(g, h, t)$
The proof of (i) and (ii) follows in a similar way.
(I_6) From $(I_8)\mu(f, f, h, t)$ is a monotonic non-decreasing function and it also tends to 1 as $t \to \infty$. Thus, $N(f, h, t)$ is a monotonic non-decreasing function and it also tends to 1 as $t \to \infty$.

Theorem 3.5 (Parallelogram Law)
Let μ be a 2-fuzzy 2-inner product on $F(X)$, for $\alpha \in (0,1)$ then α- fuzzy 2-norm induced by 2-fuzzy 2-inner product satisfies

$$\|f - g, h\|_\alpha^2 + \|f + g, h\|_\alpha^2 = 2(\|f, h\|_\alpha^2 + \|g, h\|_\alpha^2)$$

Proof :
Consider
$\|f - g, h\|_\alpha^2 + \|f + g, h\|_\alpha^2$
$= \inf\{t^2 : \mu(f - g, f - g, h, t^2) \geq \alpha\} + \inf\{s^2 : \mu(f + g, f + g, h, s^2) \geq \alpha\}$
$= \inf\{t^2 + s^2 : \mu(f - g, f - g, h, t^2) \geq \alpha, \mu(f + g, f + g, h, s^2) \geq \alpha\}$ (1)
Also,
$2(\|f, h\|_\alpha^2 + \|g, h\|_\alpha^2) = 2\{\inf\{p^2 : \mu(f, f, h, p^2) \geq \alpha\} + \inf\{q^2 : \mu(g, g, h, q^2) \geq \alpha\}\}$

$= 2\{p^2 + q^2 : \mu(f, f, h, p^2) \geq \alpha, \mu(g, g, h, q^2)\mu \geq \alpha\}$
Now (1) becomes,
$= \mu\left(f - g, f - g, h, \sqrt{2}\, p^2\right) \Lambda\, \mu\left(f + g, f + g, h, \sqrt{2} q^2\right)$
$\geq \mu(f, f, h, p^2)\ \Lambda \mu(g, g, h, q^2)$
$\therefore \|f - g, h\|_\alpha^2 + \|f + g, h\|_\alpha^2 \leq 2(\|f, h\|_\alpha^2 + \|g, h\|_\alpha^2$ (2)
Consider
$2(\|f, h\|_\alpha^2 + \|g, h\|_\alpha^2) =$
$2\left\{\left\|\frac{(f+g,h)+(f-g,h)}{2}\right\|_\alpha^2 + \left\|\frac{(g+f,h)+(g-f,h)}{2}\right\|_\alpha^2\right\}$
$\leq \frac{1}{2}\{\|f + g, h\|_\alpha^2 + \|f - g, h\|_\alpha^2 + \|g + f, h\|_\alpha^2 + \|g - f, h\|_\alpha^2\}$
$\leq \|f + g, h\|_\alpha^2 + \|f - g, h\|_\alpha^2$
$\therefore 2(\|f, h\|_\alpha^2 + \|g, h\|_\alpha^2) \leq \|f - g, h\|_\alpha^2 + \|f + g, h\|_\alpha^2$ (3)
From (2) and (3),

$$\|f - g, h\|_\alpha^2 + \|f + g, h\|_\alpha^2 = 2(\|f, h\|_\alpha^2 + \|g, h\|_\alpha^2)$$

Theorem 3.6 (Polarization Identity)
If f, g and h are elements in $F(X)$. Then
$4\, inf\{st : \mu(f, g, h, st) \geq \alpha\} = \{\|f + g, h\|_\alpha^2 - \|f - g, h\|_\alpha^2 + i\|f + ig, h\|_\alpha^2 - i\|f - ig, h\|_\alpha^2\}$

Proof :
Consider
$\{\|f+g,h\|_\alpha^2 - \|f-g,h\|_\alpha^2 + i\|f+ig,h\|_\alpha^2 - i\|f-ig,h\|_\alpha^2\}$
$= inf\{t_1^2 : \mu(f+g,f+g,h,t_1^2) \geq \alpha\}$
$-inf\{t_2^2 : \mu(f-g,f-g,h,t_2^2) \geq \alpha\}$
$+i\ inf\{t_3^2 : \mu(f+ig,f+ig,h,t_3^2) \geq \alpha\}$
$-i\ inf\{t_4^2 : \mu(f-ig,f-ig,h,t_4^2) \geq \alpha\}$
Where

$$t_1^2 = t_1'^2 + t_1''t_1''' + t_1'''t_1'' + t_1^{IV\ 2}$$
$$t_2^2 = t_2'^2 + t_2''t_2''' + t_2'''t_2'' + t_2^{IV\ 2}$$
$$t_3^2 = t_3'^2 + t_3''t_3''' + t_3'''t_3'' + t_3^{IV\ 2}$$
$$t_4^2 = t_4'^2 + t_4''t_4''' + t_4'''t_4'' + t_4^{IV\ 2}$$

$= inf\{t_1'^2 : \mu(f,f,h,t_1'^2) \geq \alpha\}$
$+inf\{t_1''t_1''' : \mu(f,g,h,t_1''t_1''') \geq \alpha\}$
$+inf\{t_1'''t_1'' : \mu(g,f,h,t_1'''t_1'') \geq \alpha\}$
$+inf\{t_1^{IV\ 2} : \mu(g,g,h,t_1^{IV\ 2}) \geq \alpha\}$
$-inf\{t_2'^2 : \mu(f,f,h,t_2'^2) \geq \alpha\}$
$+inf\{t_2''t_2''' : \mu(f,g,h,t_2''t_2''') \geq \alpha\}$
$+inf\{t_2'''t_2'' : \mu(g,f,h,t_2'''t_2'') \geq \alpha\}$
$-inf\{t_2^{IV\ 2} : \mu(g,g,h,t_2^{IV\ 2}) \geq \alpha\}$
$+i\{inf\{t_3'^2 : \mu(f,f,h,t_3'^2) \geq \alpha\}\}$
$+i\{inf\{t_3''t_3''' : \mu(f,ig,h,t_3''t_3''') \geq \alpha\}\}$
$+i\{inf\{t_3'''t_3'' : \mu(ig,f,h,t_3'''t_3'') \geq \alpha\}\}$
$+i\{inf\{t_3^{IV\ 2} : \mu(ig,ig,h,t_3^{IV\ 2}) \geq \alpha\}\}$
$-i\{inf\{t_4'^2 : \mu(f,f,h,t_4'^2) \geq \alpha\}\}$
$+i\{inf\{t_4''t_4''' : \mu(f,ig,h,t_4''t_4''') \geq \alpha\}\}$
$+i\{inf\{t_4'''t_4'' : \mu(ig,f,h,t_4'''t_4'') \geq \alpha\}\}$
$-i\{inf\{t_4^{IV\ 2} : \mu(ig,ig,h,t_4^{IV\ 2}) \geq \alpha\}\}$
Here,

$$t_1'^2 = t_2'^2 = t_3'^2 = t_4'^2 = t^2$$
$$t_1''t_1''' = t_2''t_2''' = t_3''t_3''' = t_4''t_4''' = st$$
$$t_1'''t_1'' = t_2'''t_2'' = t_3'''t_3'' = t_4'''t_4'' = ts$$
$$t_1^{IV\ 2} = t_2^{IV\ 2} = t_3^{IV\ 2} = t_4^{IV\ 2} = s^2$$
$$= inf\{st : \mu(f,g,h,st) \geq \alpha\} + inf\{ts : \mu(g,f,h,ts) \geq \alpha\}$$

$+i\{inf\{st : \mu(f,ig,h,st) \geq \alpha\}\}$
$+i\{inf\{ts : \mu(ig,f,h,ts) \geq \alpha\}\}$
$= inf\{4st : \mu(f,g,h,st) \geq \alpha\}$
$= 4\ inf\{st : \mu(f,g,h,st) \geq \alpha\}$
$\therefore 4\ inf\{st : \mu(f,g,h,st) \geq \alpha\}$
$= \{\|f+g,h\|_\alpha^2 - \|f-g,h\|_\alpha^2 + i\|f+ig,h\|_\alpha^2 - i\|f-ig,h\|_\alpha^2\}$

Theorem 3.7

$$If\ \ inf\{st : \mu(f,g,h,st) \geq \alpha\} = \frac{1}{4}\{\|f+g,h\|_\alpha^2 + \|f-g,h\|_\alpha^2 + i\|f+ig,h\|_\alpha^2 + i\|f-ig,h\|_\alpha^2\}$$

then μ is a 2-fuzzy 2-inner product on $F(X)$.

Proof :

(i) $\mu(f+g,h,|t|+|s|)=\mu((f+g),h,|t|+|s|)$

$\geq \min\{\mu(f,h,|t|),\mu(g,h,|s|)\}$

$\mu(f+g,h,|t|+|s|)\geq \min\{\mu(f,h,|t|),\mu(g,h,|s|)\}$

(ii) To prove $\mu(f,g,h,|st|)\geq$

$min\{\mu(f,f,h,|s|^2),\mu(g,g,h,|t|^2)\}$

$\mu(f,g,h,|st|)=\frac{1}{4}\{\|f+g,h\|_\alpha^2+\|f-g,h\|_\alpha^2+i\|f+ig,h\|_\alpha^2-i\|f-ig,h\|_\alpha^2\}$ (4)

Consider

$min\{\mu(f,f,h,|s|^2),\mu(g,g,h,|t|^2)\}$

$$=\min\left\{\frac{1}{4}\{\|f+f,h\|_\alpha^2+\|f-f,h\|_\alpha^2+i\|f+if,h\|_\alpha^2-i\|f-if,h\|_\alpha^2\},\frac{1}{4}\{\|g+g,h\|_\alpha^2+\|g-g,h\|_\alpha^2+i\|g+ig,h\|_\alpha^2-i\|g-ig,h\|_\alpha^2\}\right\}$$

$=\min\{\frac{1}{4}\{inf\{t\in R:\mu(f+f,f+f,h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(f+if,f+if,h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(f-if,f-if,h,t)\geq\alpha\}\}$
$+\frac{1}{4}\{inf\{t\in R:\mu(g+g,g+g,h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(g+ig,g+ig,h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(g-ig,g-ig,h,t)\geq\alpha\}\}\}$
$=\min\{\frac{1}{4}\{inf\{t\in R:\mu(f+f,f+f,h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(f+f,f+f,h,t)\geq\alpha\}\}$
$+\frac{1}{4}\{inf\{t\in R:\mu(g+g,g+g,h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(g+g,g+g,h,t)\geq\alpha\}\}\}$
$=2\|f,h\|_\alpha^2+2\|g,h\|_\alpha^2$ (5)

From (4) and (5),

$\mu(f,g,h,|st|)\geq min\{\mu(f,f,h,|s|^2),\mu(g,g,h,|t|^2)\}$

(i) $\mu(f,g,h|t|)=\frac{1}{4}\{\|f+g,h\|_\alpha^2+\|f-g,h\|_\alpha^2+i\|f+ig,h\|_\alpha^2-i\|f-ig,h\|_\alpha^2\}$
$=\frac{1}{4}\{\|g+f,h\|_\alpha^2+\|g-f,h\|_\alpha^2+i\|g+if,h\|_\alpha^2-i\|g-if,h\|_\alpha^2\}$
$=\mu(g,f,h|t|)$

(ii) To prove $\mu(\alpha f,\alpha g,h,|t|)=\mu\left(f,g,h,\frac{t}{|\alpha|^2}\right)$

$\mu(\alpha f,\alpha g,h,|t|)=\frac{1}{4}\{\|\alpha(f+g),h\|_\alpha^2+\|\alpha(f-g),h\|_\alpha^2 + i\|\alpha(f+ig),h\|_\alpha^2-i\|\alpha(f-ig),h\|_\alpha^2\}$
$=\frac{1}{4}\{inf\{t\in R:\mu(\alpha(f+g),\alpha(f+g),h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(\alpha(f-g),\alpha(f-g),h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(\alpha(f+ig),\alpha(f+ig),h,t)\geq\alpha\}$
$+inf\{t\in R:\mu(\alpha(f-ig),\alpha(f-ig),h,t)\geq\alpha\}\}$
$=\frac{1}{4}\{inf\left\{t\in R:\mu\left(f+g,f+g,h,\frac{t}{|\alpha|^2}\right)\geq\alpha\right\}$
$+inf\left\{t\in R:\mu\left(f-g,f-g,h,\frac{t}{|\alpha|^2}\right)\geq\alpha\right\}$

$+inf\left\{t \in R: \mu\left(f+ig, f+ig, h, \frac{t}{|\alpha|^2}\right) \geq \alpha\right\}$
$+inf\left\{t \in R: \mu\left(f-ig, f-ig, h, \frac{t}{|\alpha|^2}\right) \geq \alpha\right\}\}$
So that $\mu(\alpha f, \alpha g, h, |t|) = \mu\left(f, g, h, \frac{t}{|\alpha|^2}\right)$

(iii) $\mu(f, f, h, t) = 0$ for all $t \in \mathbb{C}/R^+$

(iv) $\mu(f, f, h, t) = 1$
$\Rightarrow \|f+g, h\|_\alpha^2 + \|f-g, h\|_\alpha^2 + i\|f+ig, h\|_\alpha^2 - i\|f-ig, h\|_\alpha^2 = 1$
$\Rightarrow \|f, f, h\|_\alpha^2 = 0$
(i, e) f, g, h are linearly dependent.

(v) $\mu(f, f, h, t): R \rightarrow I (= [0,1])$ is a monotonic non-decreasing function of R and $\lim \mu(f, f, t) = 1$ as $t \rightarrow \infty$, μ satisfies all the requirements and hence μ is a 2-fuzzy 2-inner product on $F(X)$.

Theorem 3.8
Let $(F(X), \mu)$ be a 2-fuzzy 2-inner product satisfying the condition that, $\mu(f, f, h, t^2) > 0$ when $t > 0$ implies $f = 0$. Then for all $\alpha \in (0,1]$, $\|f, h\|_\alpha = \inf\{t: \mu(f, f, h, t^2) \geq \alpha\}$ is a ascending family of real numbers, the 2-norm on $F(X)$. These 2-norm are called the α- 2-norms on $F(X)$ corresponding to 2-fuzzy 2-inner products.

Proof :

(i) $\|f, h\|_\alpha = 0$
$\Rightarrow \inf\{t: \mu(f, f, h, t^2) \geq \alpha\} = 0$
$\Rightarrow$for all $t \in R$, with $t > 0$, $\mu(f, f, h, t^2) \geq \alpha > 0$
$\Rightarrow f = 0$
Conversely assume that f, h are linearly dependent then
$\mu(f, f, h, t^2) = 1$ for all $t > 0$
implies that for all $\alpha \in (0,1)$, $\inf\{t: \mu(f, f, h, t^2) \geq \alpha\} = 0$.
Therefore, $\|f, h\|_\alpha = 0$

(ii) As $\mu(f, f, h, t^2)$ is invariant under any permutation it follows that $\|f, h\|_\alpha$ is invariant under any permutation.

(iii) For all α and $0 \leq p < 1$,

$$\|cf, h\|_\alpha = \inf\{t: \mu(cf, cf, h, t^2) \geq \alpha\}$$
$$= \inf\left\{t: \mu\left(f, f, h, \frac{t^2}{\|c\|}\right) \geq \alpha\right\}$$

Let $s = \frac{t}{\|c\|}$ then

$$\|cf, h\|_\alpha = \inf\{t|c|: \mu(cf, cf, h, s^2) \geq \alpha\}$$
$$= |c| \inf\{t: \mu(cf, cf, h, s^2) \geq \alpha\}$$
$$= |c|\|f, h\|_\alpha$$

(iv) $\|f, h\|_\alpha + \|g, h\|_\alpha = \inf\{t: \mu(f, f, h, t^2) \geq \alpha\} + \inf\{s: \mu(g, g, h, s^2) \geq \alpha\}$
$= \inf\{t + s: \mu(f, f, h, t^2) \geq \alpha, \mu(g, g, h, s^2) \geq \alpha\}$
$= \inf\{t + s: \mu(f+g, f+g, h, t^2 + s^2) \geq \alpha\}$
$= \|f+g, h\|_\alpha$

Hence $\|f+g, h\|_\alpha \leq \|f, h\|_\alpha + \|g, h\|_\alpha$
Thus $\{\|., .\|_\alpha, \alpha \in (0,1)\}$ is a α- 2- norm on $F(X)$.
Let $0 < \alpha_1 < \alpha_2 < 1$, then
$\|f, h\|_{\alpha_1} = \inf\{t: \mu(f, f, h, t^2) \geq \alpha_1\}$
$\|f, h\|_{\alpha_2} = \inf\{t: \mu(f, f, h, t^2) \geq \alpha_2\}$

As $\alpha_1 < \alpha_2$,
$\inf\{t: \mu(f, f, h, t^2) \geq \alpha_1\} \supset \inf\{t: \mu(f, f, h, t^2) \geq \alpha_2\}$
$\Rightarrow \inf\{t: \mu(f, f, h, t^2) \geq \alpha_2\} \geq \inf\{t: \mu(f, f, h, t^2) \geq \alpha_1\}$
$\Rightarrow \|f, h\|_{\alpha_2} \geq \|f, h\|_{\alpha_1}$
Therefore $\{\|.,.\|_\alpha, \alpha \in (0,1)\}$ is an ascending family of α- 2- norms on $F(X)$.

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