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# Isolated Signed Total Dominating Function Of Graphs

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**Abstract - An isolated signed total dominating function (ISTDF) of a graph $G$ is a function $f : V(G) \to \{-1,+1\}$ such that $\sum_{u \in N(v)} f(u) \geq 1$ for every vertex $v \in V(G)$ and for at least one vertex of $w \in V(G)$, $f(N(w)) = +1$. An isolated signed total domination number of $G$, denoted by $\gamma_{ist}(G)$, is the minimum weight of an isolated signed total dominating function of $G$. In this paper, we study some properties of ISTDF.**

***Key Words: isolated domination, signed dominating function, isolated signed dominating function, isolated signed total dominating function.***

## 1. INTRODUCTION

Throughout this paper, we consider only finite, simple and undirected graphs. The set of vertices and edges of a graph $G(p,q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$. For graph theoretic terminology, we follow [7].
For $v \in V(G)$, the open neighborhood of $v$ is $N_G(v) = \{u \in V : uv \in E(G)\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N(v)$. The degree of $v$ is $deg_G(v) = |N_G(v)|$. The minimum and maximum degree of $G$ is defined by $\delta(G) = \min_{v \in V(G)} \{deg(v)\}$ and $\Delta(G) = \max_{v \in V(G)} \{deg(v)\}$ respectively. A vertex of degree one is called a pendent vertex. A vertex which is adjacent to a pendent vertex is called a stem.
A function $f : V(G) \to \{0,1\}$ is called a dominating function if for every vertex $v \in V(G)$, $f(N[v]) \geq 1$ [8]. The weight of $f$, denoted by $w(f)$ is the sum of the values $f(v)$ for all $v \in V(G)$. Various domination functions has been defined from the definition of dominating function by replacing the co-domain $\{0,1\}$ as one of the sets $\{-1,0,1\}$, $\{-1,+1\}$ and etc. One of such example is signed dominating function [3, 4].
In 1995, J.E.Dunbar et al. [4] defined signed dominating function. A function $f : V(G) \to \{-1,+1\}$ is a signed dominating function of $G$, if for every vertex $v \in V(G)$, $f(N[v]) \geq 1$. The signed domination number, denoted by $\gamma_s(G)$, is the minimum weight of a signed dominating

function on $G$ [4]. The signed dominating function has been studied by several authors including [1, 2, 5, 6, 9, 10].

A subset $S$ of vertices of a graph $G$ is a total dominating set of $G$ if every vertex in $V(G)$ has a neighbor in $S$. The minimum cardinality of a total dominating set of $G$ is said to be the total domination number and is denoted by $\gamma_t(G)$. A subset $S$ of vertices of a graph $G$ is a 2-total dominating set of $G$ if every vertex in $V(G)$ has at least two neighbors in $S$. The minimum cardinality of a 2total dominating set of $G$ is said to be the total domination number and is denoted by $\gamma_{2,t}(G)$.

In 2016, Hameed and Balamurugan [11] introduced the concept of isolate domination in graphs. A dominating set $S$ of a graph $G$ is said to be an isolate dominating set if $< S >$ has at least one isolated vertex [11]. An isolate dominating set $S$ is said to be minimal if no proper subset of $S$ is an isolate dominating set. The minimum and maximum cardinality of a minimal isolate dominating set of $G$ are called the isolate domination number $\gamma_0(G)$ and the upper isolate domination number $\Gamma_0(G)$ respectively.

By using the definition of signed total dominating function and isolate domination, we introduced the concept of isolated signed total dominating function. An isolated signed total dominating function (ISTDF) of a graph $G$ is a function $f : V(G) \to \{-1,+1\}$ such that P $f(u) \geq 1$ for every vertex $v \in V(G)$ and for at
$u \in N(v)$

least one vertex $w \in V(G)$, $f(N(w)) = +1$. An isolated signed total domination number of $G$, denoted by $\gamma_{ist}(G)$, is the minimum weight of an isolated signed total dominating function of $G$. In this paper, we study some properties of ISTDF and we give isolated signed total domination number some classes of graphs.

## 2. MAIN RESULTS

**Lemma 1.** *Let G be any graph in which* $deg(v)$ *is even for all* $v \in V(G)$. *Then G does not admit ISTDF.*

*Proof.* Note that $|N(u)|$ is even for any vertex $u \in V(G)$. Thus there exist no vertex $u \in V(G)$ such that $f(N(u)) = 1$ for any function $f : V(G) \to \{-1,+1\}$. □

**Lemma 2.** *For any graph G which admits ISTDF,* $\gamma_{st}(G) \leq \gamma_{ist}(G)$.

*Proof.* Since every ISTDF is a STDF, it follows that $\gamma_{st}(G) \leq \gamma_{ist}(G)$. □

In [12], Bohdan Zelinka and Liberec proved the following result which gives an lower bound for STDN of regular graphs.

**Theorem 3.** *[12] Let G be a regular graph of degree r. If r is odd, then* $\gamma_{st}(G) \geq \frac{n}{r}$; *if r is even, then* $\gamma_{st}(G) \geq \frac{2n}{r}$.

**Theorem 4.** *Let G be an odd regular graph of degree* $r(\geq 3)$, *then* $\gamma_{ist}(G) \geq \frac{n}{r}$.

*Proof.* Let $u \in V(G)$ and $r = 2` + 1$ for some integer $` \geq 1$. Define a function $f : V(G) \to \{-1,+1\}$ by labeling any of the ` neighbors of $u$ by $-1$ sign and all the remaining vertices of $G$ by +1 sign. Then $f$ is an ISTDF and $f(N(u)) = 1$. This means that any odd regular graph of degree $r(\geq 3)$ must admits ISTDF. Thus from Lemma 2 and theorem 3, we can have the result. □

**Lemma 5.** *When n is even,* $\gamma_{ist}(K_n) = 2$ *for n even.*

*Proof.* By Theorem 4, $\gamma_{ist}(G) \geq \frac{n}{r} = \frac{n}{n-1}$. This means that $\gamma_{ist} > 1$ and so $\gamma_{ist} \geq 2$.

Define a function $f : V(G) \to \{-1,+1\}$ by labeling any of the $\frac{n}{2}+1$ vertices of $G$ by +1 sign and all the remaining $\frac{n}{2}-1$ vertices of $G$ by $-1$ sign. Then $f$ is an ISTDF and $f(N(u))$

= 1 for any vertex which received the label +1. Also $w(f) = (+1)\left(\frac{n}{2}+1\right)+(-1)\left(\frac{n}{2}-1\right) = 2$ ans so $\gamma_{ist}(G) \le 2$. □

**Theorem 6.** *Let $n \ge 2$ be an integer and let $G$ be a disconnected graph with $n$ components $G_1, G_2, \ldots, G_n$ such that the first $r(\ge 1)$ components $G_1, G_2, \ldots, G_r$ admit ISTDF. Then* $\gamma_{ist}(G) = \min_{1\le i\le r} \{t_i\}$, *where* $t_i = \gamma_{ist}(G_i) + \sum_{j=1, j\ne i}^{n} \gamma_{st}(G_j)$.

*Proof.* Assume that $t_1 = \min_{1\le i\le r} \{t_i\}$.

Let $f_1$ be an minimum ISTDF of $G_1$ and $f_i$ be a minimum STDF of $G_i$ for each $i$ with $2 \le i \le n$. Then $f : V(G) \to \{-1,+1\}$ defined by

$$f(x) = f_i(x),\ x \in V(G_i),$$

is an ISTDF of $G$ with weight $\gamma_{ist}(G_1) + \sum_{i=2}^{n} \gamma_{st}(G_i)$ and so $\gamma_{ist}(G) \le \gamma_{ist}(G_1) + \sum_{i=2}^{n} \gamma_{st}(G_i) = t_1$.

Let $g$ be a minimum ISTDF of $G$. Then there exists an integer $j$ such that $g|_{G_j}$ is a minimum ISTDF of $G_j$ for some $j$ with $1 \le j \le r$.

Also for each $i$ with $1 \le i \le n (i \ne j)$, $g|_{G_i}$ is a minimum STDF of $G_i$. Therefore $w(g) \ge \gamma_{ist}(G_j) + \sum_{i=1, i\ne j}^{n} \gamma_{st}(G_i) = t_j \ge t_1$ and hence $\gamma_{ist}(G) = \min_{1\le i\le r} \{t_i\}$. □

**Corollary 7.** *Let $H$ be any graph which does not admit ISTDF. Then $G = H \cup rK_2 (r \ge 1)$ admits ISTDF with $\gamma_{ist}(G) = 2r + \gamma_{st}(H)$*

*Proof.* Let $G_i \cong K_2$ for $1 \le i \le r$ and $G_{r+1} \cong H$. Note every vertex of each copy of $K_2$ receive the label +1. Thus by Theorem 6, we have $\gamma_{ist}(G) = 2r + \gamma_{st}(H)$. □

**Lemma 8.** *Let $f$ be an ISTDF of $G$ and let $S \subset V$. Then $f(S) = |S| (mod\ 2)$.*

*Proof.* Let $S^+ = \{v | f(v) = 1, v \in S\}$ and $S^- = \{v | f(v) = -1, v \in S\}$. Then $|S^+| + |S^-| = |S|$ and $|S^+| - |S^-| = f(S)$. If both $S^-$ and $S^+$ are either odd or even, then both $|S|$ and $f(S)$ must be even.

If either one of $S^-$ and $S^+$ is odd and another one is even, then both $|S|$ and $f(S)$ must be odd. Therefore $f(S) = |S| (mod\ 2)$. □

**Lemma 9.** *Let $G$ be a graph of order $n$ and $\delta \ge 2$. Then* $2\gamma_{2,t}(G) - n \le \gamma_{ist}(G)$.

*Proof.* Let $g$ be a minimum isolate signed total dominating function of $G$. Let $V^+ = \{u \in V : g(u) = +1\}$ and $V^- = \{v \in V : g(v) = -1\}$. If $V^- = \varphi$, then the proof is clear.

Suppose there exists a vertex $v \in V^-$. Since $g(N(v)) \ge 1$ and $\delta \ge 2$, then $v$ has at least two adjacent vertices in $V^+$. In the similarly manner, if $v \in V^+$, then $v$ has at least two adjacent vertices in $V^+$.

Therefore $V^+$ is a 2-total dominating set for $G$ and so $|V^+| \ge \gamma_{2,t}(G)$. Since $\gamma_{ist}(G) = |V^+| - |V^-|$ and $n = |V^+| + |V^-|$, we have $\gamma_{ist}(G) = 2|V^+| - n$ and so $\gamma_{ist}(G) \ge 2\gamma_{2,t}(G) - n$. □

**Remark 10.** (a) Let $G$ be a graph which admits a 2-total dominating set $S$. Then $N(v) \subseteq S$ whenever $|N(v)| = 2$ for any vertex $v \in V(G)$.

(a) Let $G$ be a graph which admits an ISTDF function(or STDF), say $f$. Then the vertices of $N(v)$ are labeled with +1 sign whenever $|N(v)| \le 2$ for any vertex $v \in V(G)$.

**Remark 11.** The inequality given in Lemma 9 is sharp. For example, consider the following graph $G$.

Every 2-total dominating set contain the vertices 2, 4, 6 and 8(by Remark 10(a)). Thus $\gamma_{2,t}(G) \geq 4$. Also $\{2,4,6,8\}$ is a 2-total dominating set and so $\gamma_{2,t}(G) \leq 4$.
For every ISTDF $f$ of $G$, it is true that $f(2) = f(4) = f(6) = f(8) = +1$(by Remark 10(b)). Thus $\gamma_{ist}(G) \geq 0$. Now label the vertices of $G$ by $g(2) = g(4) = g(6) = g(8) = +1$ and $g(1) = g(3) = g(5) = g(7) = -1$. Then $g$ is a ISTDF with $w(g) = 0$ and so $\gamma_{ist}(G) \leq 0$. Thus for the graph $G$, we have $2\gamma_{2,t}(G) - n = 2(4) - 8 = 0 = \gamma_{ist}(G)$.
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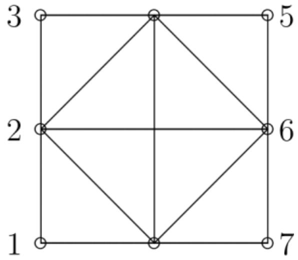

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Figure 1: $G$

**Lemma 12.** *The complete bipartite graph $K_{m,n} = (A,B)$ admits ISTDF if and only if, either $m$ or $n$ is odd.*

*Proof.* Suppose $K_{m,n} = (A,B)$ admits ISTDF, say $f$. On the contrary, suppose both $m$ and $n$ are even. In this case, for every vertex $v \in V(A)$, $f(N(v)) = f(B) \geq 2$(Since $n$ is even). Also for every vertex $v \in V(B)$, $f(N(v)) = f(A) \geq 2$(Since $m$ is even). Thus there does not exist a vertex such that $f(N(v)) = 1$, a contradiction.
Conversely suppose either $m$ or $n$ is odd. With out loss of generality, assume that $m$ is odd. Define s function $f : V(G) \to \{+1,-1\}$ as follows. Label any $\frac{m+1}{2}$ vertices of $A$( $\lceil \frac{n+1}{2} \rceil$ vertices of $B$) by +1 sign and label the remaining $\frac{m-1}{2}$ vertices($\lfloor \frac{n-1}{2} \rfloor$ vertices of $B$) of $A$ by -1 sign. Then $f$ is a SDF. Also $f(N(v)) = (\frac{m+1}{2})(+1) + (\frac{m-1}{2})(-1) = 1$ for all $v \in B$. □

**Remark 13.** *It is proved in the above lemma that the ISTDF for the complete bipartite graph $K_{m,n}$ does not exist when both m and n are even. When it admits ISTDF, the ISTD number is given by*
$\gamma_{ist} = 2$ *when both m and n are odd; and 3 if either m or n is even.*

**Theorem 14.** *For given integer $k \geq 1$, there exists a graph G such that $\gamma_{st}(G) = \gamma_{ist}(G) = k$.*

*Proof.* Let $G$ be a graph such that $V(G) = \{a_1,a_2,...,a_{2k}, b_2, b_4, b_6, ..., b_{2k}\}$ and $E(G) = \{a_ia_{i+1}/1 \leq i \leq 2k-1\} \cup \{a_{2k}a_1\} \cup \{a_{2i}b_{2i} : 1 \leq i \leq k\}$.
Let $f$ be a ISTDF of $G$. Then by Remark 10(b), $f(a_i) = +1$ for all $i$ with $1 \leq i \leq 2k$. Thus $f(V(G)) \geq 2k(+1) + k(-1) = k$ and so $\gamma_{st}(G) \geq k$.
Define a function $g : V(G) \to \{-1,+1\}$ by $g(a_i) = +1$ and $g(b_i) = -1$. Then $g$ is a STDF such that $w(f) = k$ and $f(N(b_2)) = 1$. Therefore $\gamma_{ist}(G) \leq k$. Since $\gamma_{st}(G) \leq \gamma_{ist}(G)$, we have $\gamma_{st}(G) = \gamma_{ist}(G) = k$. □

**Lemma 15.** *If $G = mK_2 \cup B$, where B is a graph which is an union of cycles($m \geq 1$ and B may be empty), then $\gamma_{ist}(G) = n$.*

*Proof.* Let $f$ be an ISTDF of $G$ and $u \in V(G)$.
**Case 1:** If $u \in V(mK_2)$, then by Remark 10(b), $f(u) = +1$. **Case 2:** If $u \in V(B)$ then $|N(u)| = 2$ and so by Remark 10(b), $f(u) = +1$. Thus $w(f) = n$ and so $\gamma_{ist}(G) \geq n$. But always $\gamma_{ist}(G) \leq n$ and so $\gamma_{ist}(G) = n$. □

**Remark 16.** The converse of the above result is not true. Consider the following graph $G$. From Remark 10(b), $f(u) = +1$ for any vertex $u \in V(G)$ and for any ISTDF $f$. Thus $\gamma_{ist}(G) \geq n$. But always $\gamma_{ist}(G) \leq n$ and so $\gamma_{ist}(G) = n$.

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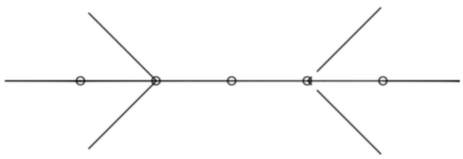

b b b Figure 2: *G* b b

**Remark 17.** *Let G be a graph of order n which admits ISTDF. Then* $\gamma_{is}(G)$ 6= $n - 1$.

*Proof.* Let $f$ be a minimum ISTDF of $G$. Suppose $f(u) = +1$ for all $u \in V(G)$, then $\gamma_{ist}(G) = n$. Suppose $f(u) = -1$ for some $u \in V(G)$, then $\gamma_{ist}(G) \leq n - 2$. □

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