International Journal of Aquatic Science
ISSN: 2008-8019
Vol 12, Issue 02, 2021

# An Application Of Max-Radial Number Of Graphs In Game Theory

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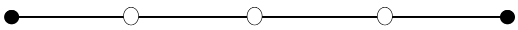
***Abstract :*** ***For a graph $G(V, E)$, the $S$-radial set, $B_R(S)$, is defined for any set $S \subseteq V$, as the set of vertices $u \in V \backslash S$ which are at a distance of radius of $G$ from some vertex $v \in S$. The Max-Radial number of $G$ is the parameter which is defined as $\max_S\{|B_R(S)| - |S|\}$. The study on this parameter faces the challenge of placing the maximum number of maximal length strings with certain conditions in any graph model. In this paper, we study the varied properties of this parameter. We characterize the extremal graphs for the Max-Radial concept in graphs. Also we prove the existence of graphs with given order and Max-Radial number.***
***Keywords: Metrics, Differential, Max-Radial number, R-Differential, Radius, Diameter.***
***AMS Subject Classification code: 05C(Primary)***

## 1. INTRODUCTION

Let us first introduce the following game, which is a two person zero sum game. Player X will provide a graph model $G$ of order $n$ and a set of boxes $M_1, M_2, \dots M_n$ where $M_i$ contains enough number of strings of length $i$ with a red end R and a blue end B with $i - 1$ flexible joints in between them (isomorphic to path $P_{i+1}$). A string is depicted here for more clarity.



R $M_4$ - String B

Now player Y has to select a box $M_l$ such that $l$ should be maximum with the property that if R is fixed at any vertex of $G$, then the string must fit a path in $G$. Player Y gains Rs 2 if he succeeds in choosing such $l$, otherwise he loses Rs 2 to Player X and the game continues after Player X reveals the value of $l$.

Next Player Y has to select a set $S$ of vertices in $G$. He pays Re 1 for each distinct of $S$ to Player X. Then he has to fix the R end of strings of $M_l$ with the following conditions.
(i) Any number of R end of $M_l$ strings can be fixed at a vertex of $S$.
(ii) Each string must fit with any $uv$ geodesic path of $G$ where $u \in S$ & $v \in G \backslash S$.

For each distinct vertex of $G\backslash S$ at which B is fixed, Player Y receives Rs 1 from Player X.

Mathematically speaking, maximum profit of Player Y=Max-Radial number of $G$ + 2. It is clear that if Player Y choses $l$ to be the radius of $G$, then he wins Rs 2.

Let us recall the basic terminology before defining the Max-Radial number.

In this paper, the graphs considered are *connected*, *undirected* and *finite graphs*. For other notation and terminology, we follow [5, 8]. Let $G$ be a graph of order $n$. A vertex $v \in V(G)$ is called a *full vertex*, if it is adjacent to all other vertices. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

The *distance* $d(u,v)$ between two vertices $u$ and $v$ in G is the length of a shortest path joining them. The *eccentricity* $e(v)$ of a vertex $v$ in a connected graph G is maximum distance $d(u,v)$ for all $u$ in $G$. The *radius* $rad(G)$ is the minimum eccentricity of the vertices. The *diameter* $d(G)$ is the maximum eccentricity of the vertices. For every two vertices $u$ and $v$ in a graph $G$, a $u-v$ *geodesic* is a shortest path between $u$ and $v$. For further reference on distance in graphs, one can refer [6].
For a subset $S$ of the vertex set $V(G)$, let $< S >$ denote the *induced subgraph* of $G$ induced by $S$. For any vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set of all vertices adjacent to $v$. That is, $N(v) = \{u \in V(G): uv \in E(G)\}$. The *closed neighborhood* $N[v]$ of $v$ is defined by $N[v] = N(v) \cup \{v\}$. An $n$ *-factor* of $G$ is a $n$-regular spanning subgraph of $G$.
We present the following game in literature which inspired us to work on Max-Radial number of a graph. We are allowed to buy as many tokens as we like, at a cost of $1 each. For example, suppose that we buy $k$ tokens. We then place the tokens on some subset of $k$ vertices of $G$. For each vertex of $G$ which has no token on it, but is adjacent to a vertex with a token on it, we receive $1. Our objective is to maximize our profit, that is, the total value received minus the cost of the tokens bought. Notice that we do not receive any credit for the vertices on which we place a token.
The definition of the A-differential of a set was first given by Mc Rae and Parks, while the definition of $\partial(S)$ was given by S.T Hedetniemi about ten years ago [12] . The parameter $\partial(S)$ is also considered by Goddard and Henning [7], who denoted $\partial(S)$. The minimum differential of an independent set has been considered by Zhang [13], who showed that this parameter can be computed in polynomial time.

For a set $S \subseteq V$, we define: $I(S) = S - N(S)$, the isolates in $\langle S \rangle$, the vertices in $S$ having no neighbors in $S$,
$A(S) = S \cap N(S)$, the non-isolates in $\langle S \rangle$, the vertices in $S$ having a neighbor in $S$,
$B(S) = (V - S) \cap N(S)$, the boundary of $S$, the vertices in $V - S$ dominated by $S$.

It is easy to observe that for a disconnected graph $G$ with components $G_1, G_2, G_3, \ldots, G_n$, $\partial(G) = \partial(G_1) + \partial(G_2) + \partial(G_3) + \cdots + \partial(G_n)$.
Motivated by this concept, KM. Kathiresan and M. Mathan [9] introduced the concept of R-differential in graphs by taking $S$-radial set in place of boundary of $S$. For a set $S \subseteq V$, we define $S$-radial set $B_R(S)$ to be the set of vertices $u \in V\backslash S$ such that $d(u,v) = r$ for some $v \in S$. The R-differential of a set S is defined as $\partial_R(S) = |B_R(S)| - |S|$. The R-differential of the graph $G$ is $max\{\partial_R(S) : S \subseteq V\}$. The R-differential is denoted by $\partial_R$.
We call R-differential as Max-Radial number to avoid ambiguity arising out of the resemblance of the former term with other familiar mathematical concept.

In this paper, we prove some basic results on the Max-Radial number in graphs. We characterize the extremal graphs for the Max-Radial number concept in graphs. Also we prove the existence graphs with given order and Max-Radial number.

## 2. MAIN RESULTS

From the definition of Max-Radial number in graphs, the following facts can be easily verified.

**Fact 2.1** For a complete graph $K_n$ , $\partial_R(K_n) = n - 2$.

**Fact 2.2**

For any path $P_n$, $\partial_R(P_n) = \begin{cases} 1 & if\ n\ is\ odd \\ 0 & if\ n\ is\ even\,. \end{cases}$

**Fact 2.3** For any cycle $C_n$, $n \geq 3$,

$$\partial_R(C_n) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & if\ n\ is\ odd \\ 0 & if\ n\ is\ even\,. \end{cases}$$

**Fact 2.4** For the star graph $K_{1,n}$ **,** $\partial_R(K_{1,n}) = n - 2$.

**Fact 2.5** For the complete bipartite graph $K_{m,n}$ **,** $\partial_R(K_{m,n}) = m + n - 4$, $m \geq 2$ and $n \geq 2$.

**Fact 2.6** For a wheel graph $W_n$, $n \geq 4$, $\partial_R(W_n) = n - 2$.

**Fact 2.7** $\partial_R(G) < n - 1$ for any graph $G$.
Since $\neq \emptyset$ , $B_R(X) \subset V - X$. Therefore $|B_R(X)| < n - 1$.

**Fact 2.8** If $X$ is a $\partial_R - set$ of $G$, then $\partial_R(G) = 0$ if and only if $|B_R(X)| = |X|$

Therefore, for any graph $G$, $\partial_R(G)$ varies between 0 and $n - 2$.

Let us first introduce the definition of radial graph before proceeding the theorem.

Two vertices of a graph are said to be *radial* to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph $G$, denoted by $R(G)$, has the vertex set as in $G$ and two vertices are adjacent in $R(G)$ if and only if they are radial in $G$. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of $G$. A graph $G$ is called a *radial graph* if $R(G) = G$ for some graph $H$.

Further details on radial graph one can refer [1,10]. The following results have been proved in [10].

**Result 2.9** Let $P_n$ be any path on $n \geq 5$ vertices, then $R(P_n) = \begin{cases} \left(\frac{n}{2}\right) K_2 & if\ n\ is\ even \\ P_3 \cup \left(\frac{n-3}{2}\right) K_2 & if\ n\ is\ odd \end{cases}.$

**Result 2.10** Let $C_n$ be any cycle on $n \geq 4$ vertices, then

$$R(C_n) = \begin{cases} \left(\frac{n}{2}\right) K_2 & if\ n\ is\ even \\ \cong C_n & if\ n\ is\ odd \end{cases}.$$

**Result 2.11** $R(K_{m,n}) = K_m \cup K_n$

**Result 2.12** For a graph $G$ of order $n$, $R(G) = K_n$ if and only if either $G$ or $\bar{G}$ is $K_n$.

**Result 2.13** If $r(G) > 1$, then $R(G) \subseteq \bar{G}$.

**Result 2.14** Every graph $G$ of order $n$ with $\Delta(G) = n - 1$ is a radial graph of itself.

Next we prove some characterization theorems. We note that Radial graph $R(G)$ of a connected graph $G$ contains no isolated vertices.

**Theorem 2.15**

For any graph $G$, $\partial_R(G) = 0$ if and only if $R(G) \cong F$ where $F$ is 1-factor.

***Proof***

Let $G$ be a graph on n vertices with $\partial_R(G) = 0$.
Let $R(G)$ be the radial graph of $G$.
Suppose $R(G) \ncong F$ then there exists a vertex $u \in V(R(G))$ such that $\deg(u) \geq 2$ .
This implies that $u$ has atleast two vertices at a distance $r$ in $G$.
When $X = \{u\}$, $|B_R(X)| - |X| \geq 1$. Therefore, $\partial_R(G) = \max\{\partial_R(X) : X \subseteq V(G)\} \geq 1$ which is contradiction. Thus $R(G) \cong F$.
Conversely, suppose that $R(G) \cong F$.
Claim: $\partial_R(G) = 0$.

Since $R(G)$ is 1-factor, then $R(G) = K_2 \cup K_2 \ldots \ldots \cup K_2$ $(\frac{n}{2}\ times)$.

$$\begin{aligned} \text{Now } \partial_R(G) &= \partial(R(G)) \\ &= \partial(K_2 \cup K_2 \ldots \ldots \cup K_2) \\ &= \partial(K_2) + \partial(K_2) + \cdots + \partial(K_2) \\ &\qquad\qquad (\tfrac{n}{2}\ times) \\ &= \frac{n}{2}[\partial(K_2)] \\ &= 0 \end{aligned}$$

That is, $\partial_R(G) = 0$. Hence the proof. □

**Theorem 2.16**

For any graph $G$ of order $n$ , $\partial_R(G) = n - 2$ if and only if $G$ contains a full vertex.

***Proof*** Let G be a graph of order n with $\Delta = n - 1$.

$$\begin{aligned} \text{Now, } \Delta = n - 1 &\Leftrightarrow rad(G) = 1 \\ &\Leftrightarrow \partial_R(G) = \partial(G) \\ &\Leftrightarrow \partial_R(G) = n - 2 \end{aligned}$$

Hence $\Delta = n - 1 \Leftrightarrow \partial_R(G) = n - 2$. □

We now construct some graphs with required Max-radial number. The following theorem constructs a graph $G$ with $\partial_R(G) = m$ for any given natural number $m$.

**Theorem 2.17**

For any given natural number $m$, there exists a graph $G$ such that $\partial_R(G) = m$.

***Proof***

Let $m$ be any given natural number.

***Case (i)*** Suppose $m \geq 1$ is odd.

Construct a graph $G$ with $V(G) = \{v_1, v_2, v_3;\ w_1, w_2, \dots w_k; u_1, u_2, \dots u_{m+1-k}\}$ where $1 \leq k \leq m+1$ and $E(G) = \{v_1 w_i, v_1 v_2, v_2 v_3, v_3 u_j\ :\ 1 \leq i \leq k, 1 \leq j \leq m+1-k\}$ . Now we claim that $\partial_R(G) = m$. Let $S = \{v_2\} \subseteq V(G)$. Then $B_R(S) = \{w_1, w_2, \dots . w_k\ ;\ u_1, u_2, \dots . u_{m+1-k}\}$. Therefore, the Max-Radial number of the set $S$ is $\partial_R(S) = |B_R(S)| - |S| = (m+1) - 1 = m$. Thus $\partial_R(G) \geq m$.

It is enough if we prove $\partial_R(G) \leq m$. On contrary assume that $\partial_R(G) > m$. Suppose there exists a Max-Radial set $S \subseteq V(G)$ containing atleast two vertices. Then $B_R(S)$ must contain atleast $m+3$ vertices to have $\partial_R(G) > m$ which is a contradiction to the fact that, the order of $G$ is $m+4$. Therefore, $S$ must be a singleton set. Also for any other singleton set $S'$ other than $\{v_2\}$ we have $\partial_R(G) < m$. Therefore, $S = \{v_2\}$ is the unique Max-Radial set of $G$. Hence $\partial_R(G) = m$.

***Case (ii)*** Suppose $m \geq 2$ is even.

Construct a graph $G$ with $V(G) = \{v_1, v_2, v_3, v_4\ ; w_1, w_2, \dots w_k; u_1, u_2, \dots\ u_{m+2-k}\}$ where $1 \leq k \leq m+2$ and $E(G) = \left\{v_1 w_i, v_1 v_2, v_2 v_3, v_3 v_4, v_4 u_j : 1 \leq i \leq k,\ 1 \leq j \leq m+2-k\right\}$. Now we claim that $\partial_R(G) = m$. Let $S = \{v_2, v_3\} \subseteq V(G)$. Then $B_R(S) = \{u_1, u_2, \dots . u_{m+2-k}\ ;\ w_1, w_2, \dots . w_k\}$. Therefore, the Max-Radial number of the set $S$ is $\partial_R(S) = |B_R(S)| - |S| = (m+2) - 2 = m$. Thus $\partial_R(G) \geq m$.

Next we prove that $\partial_R(G) \leq m$. On contrary assume that $\partial_R(G) > m$. Suppose a subset $S \subseteq V(G)$ contains more than two vertices. Then $B_R(S)$ must contain atleast $m+4$ vertices to have $\partial_R(G) > m$, which is a contradiction, since $V(G)$ which is the union of $S$ and $B_R(S)$ has $m+5$ vertices only. Therefore, $S$ must contain atmost two vertices. We can note that for any other set $S'$ containing atleast two vertices other than $\{v_2, v_3\}$, $\partial_R(G) < m$. In addition, no singleton set $S'$ has $B_R(S')$ with more than $m$ vertices. Therefore, $\partial_R(G) = m$. □

**Theorem 2.18**

For any two positive integers $m$ and $n$, $n \geq 3$ and $1 \leq m \leq n-2$, there exists a graph $G$ of order $n$ and max-radial number $m$.

***Proof***

Let $m$ and $n$ be two positive integers, $n \geq 3$ and $1 \leq m \leq n-2$.

***Case (i)*** Suppose $n - m$ is even.

Construct a graph $G$ with $V(G) = \{v_1, v_2, v_3, \dots . v_{\left\lfloor\frac{n-m}{2}\right\rfloor}, v_{\left\lfloor\frac{n-m}{2}\right\rfloor+1}, \dots . v_{n-m-1}, v_{n-m}, w_1, w_2, \dots .\ w_m\}$ and $E(G) = \{v_i v_{i+1}\ /\ 1 \leq i \leq n-m-1\} \cup \left\{v_{n-m-1} w_j, w_j v_{n-m}\ /\ 1 \leq j \leq m\right\}$. We claim that $\partial_R(G) = m$. Let $S = \left\{v_{\left\lfloor\frac{n-m}{2}\right\rfloor}\right\} \subseteq V(G)$.Then $B_R(S) = \{w_1, w_2, \dots w_m, v_{n-m}\}$. Therefore, the Max-Radial number of the set $S$ is $\partial_R(S) = |B_R(S)| - |S| = (m+1) - 1 = m$. Thus $\partial_R(G) \geq m$. Suppose a set $S$ contains at most two vertices. Then $B_R(S)$ contains at most $m+1$ vertices. $\partial_R(S) = |B_R(S)| - |S| < m$. Thus $\partial_R(G) < m$. Therefore, $S = \{v_{\left\lfloor\frac{n-m}{2}\right\rfloor}\}$ is only $\partial_R - set$ of $G$ with $\partial_R(S) = m$. Hence $\partial_R(G) = m$.

***Case (ii)*** Suppose $n - m$ is odd.

Construct a graph $G$ with $V(G) = \{v_1, v_2, v_3, \dots . v_{n-m-1}, w_1, w_2, \dots w_m, w\}$ and $E(G) = \{v_i v_{i+1}, v_{n-m-1} w_j, w v_{\left\lfloor\frac{n-m+1}{2}\right\rfloor+1},\ w v_{\left\lfloor\frac{n-m+1}{2}\right\rfloor-1}\ /1 \leq i \leq n-m-2\ and\ 1 \leq j \leq m\}$. we prove that $\partial_R(G) = m$. Let $S = \{v_{\left\lfloor\frac{n-m+1}{2}\right\rfloor}\} \subseteq V(G)$.Then $B_R(S) = \{v_1, w_1, w_2,$

$\ldots w_m\}$, $\partial_R(S) = (m+1) - 1 = m$. Therefore, $\partial_R(G) \geq m$. It is enough to prove that $\partial_R(G) \leq m$. Suppose $S$ contains atleast two vertices. Then $B_R(S)$ contains at most $m+1$ vertices, $\partial_R(G) < m$. Therefore, $S = \{v_{\lfloor \frac{n-m+1}{2} \rfloor}\}$ is only $\partial_R - set$ of $G$ with $\partial_R(S) = m$. Hence $\partial_R(G) = m$. This complete the proof.

**For example,**

(i) when $n = 9, m = 5$ in Theorem 2.18, the constructed graph is shown in Figure 2.1.

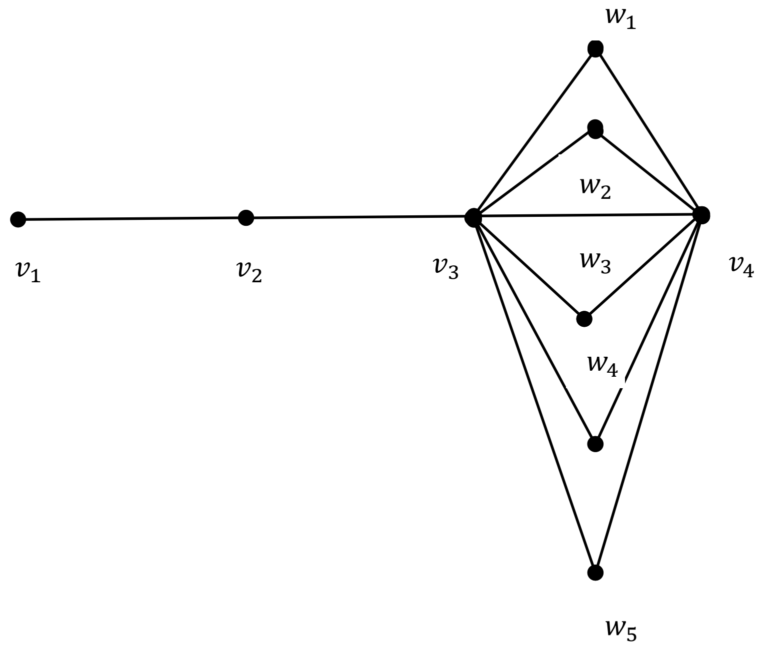



**Figure 2.1**

Here $\partial_R(G) = 5$.

(ii) when $n = 8, m = 3$ in Theorem 2.18, the constructed graph is shown in Figure 2.2.

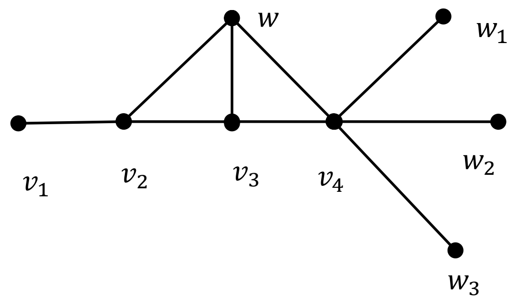



**Figure 2.2**

Here $\partial_R(G) = 3$.

**Theorem 2.19**

Given a graph $H$ and any integer $m > 0$, there exists a graph $G$ such that $H$ is an induced subgraph of $G$ and $\partial_R(G) - \partial_R(H) = m$.

***Proof***

Let $H$ be a given graph on $n$ vertices and $m$ be the positive integer.

Suppose $V(H) = \{v_1, v_2, \ldots . v_n\}$.

**Case (i)** $H$ has a full vertex $v$.

Then $\Delta(H) = n - 1$ and therefore, $\partial_R(H) = n - 2$. Construct a graph $G$ from the graph $H$ by attaching $m$ pendent vertices $w_1, w_2, \ldots . w_m$ at $v$. Obviously $H$ is an induced subgraph of $G$.

Take $S = \{v\} \subseteq V(G)$. Then $B_R(S) = \{v_2, v_3, \ldots . v_n, w_1, w_2, \ldots . w_m\} = V(G) - \{v\}$. Clearly $v$ is a full vertex in $G$, we have the max-radial number of $G$ is $\partial_R(G) = (m + n) - 2$. Now $\partial_R(G) - \partial_R(H) = \big((m + n) - 2\big) - (n - 2) = m$, implies that $\partial_R(G) - \partial_R(H) = m$.

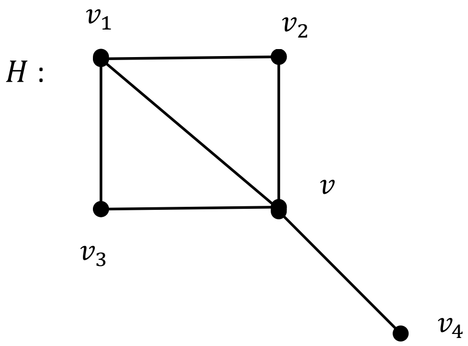
**Case(ii)** $H$ has no full vertex $v$ with radius $r$.

Let $S_1$ be a minimum $\partial_R - set$ of $H$ and $\partial_R(H) = t$. Then $B_R(S_1)$ contains $t + s$ vertices for some positive integer s. Let $v \in B_R(S_1)$. Then $d(u, v) = r$ for some $u \in S_1$. Consider $u - v$ geodesic $P: uv_1v_2 \ldots . v$. Construct a graph $G$ from $H$ by attaching $m$ new vertices $w_1, w_2, \ldots . , w_m$ & edges $\{w_iw \,/\, w \in N[v], 1 \leq i \leq m\}$. Clearly $H$ is an induced subgraph of $G$. In addition, $d(u, w_i) = r$ for all $i$, $1 \leq i \leq m$. Since all $w_i$'s are isomorphic images of $v$, $rad(G) = rad(H) = r$. $S_1$ serves to be a minimum $\partial_R - set$ of $G$, also In $G$, $B_R(S_1)$ also contains $\{w_i \,/\, 1 \leq i \leq m\}$ along with $B_R(S_1)$ set of $H$. Hence $|B_R(S_1)| = s + t + m$. $\partial_R(G) = m + t$ making $\partial_R(G) - \partial_R(H) = m$. ■

This complete the proof. □

As an illustration for given $H$ and any integer $m > 0$ in Theorem 2.19, the constructed graph $G$ with $\partial_R(G) = m + t$, is shown in Figure 2.3 and 2.4.

**Case (i):** $H$ has a full vertex $v$ and $m = 5$.



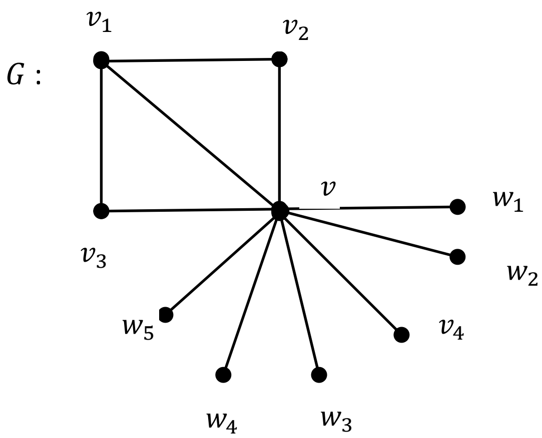
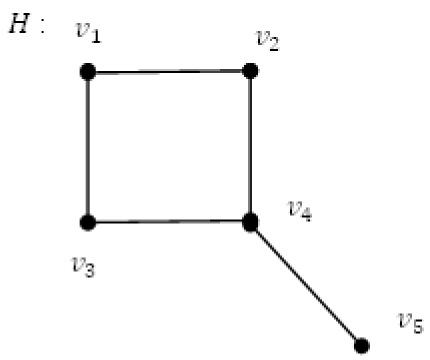




**Figure 2.3**

Here $\partial_R(H) = 3$ and $\partial_R(G) = 5$.

**Case (ii)** $H$ has no full vertex $v$ and radius 2. Let $m = 3$.

Consider the geodesic $P$: $v_5v_4v_2$ in Figure 2.4.



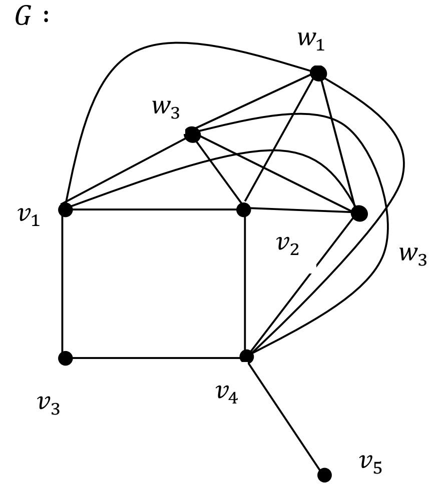
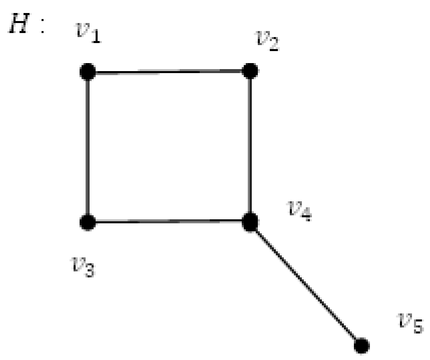



**Figure 2.4**

Here $S_1 = \{v_5\}$ is a minimum $\partial_R - set$ of $H$, $\partial_R(H) = 1$ and $\partial_R(G) = 4$.

**Note** The above constructed graph $G$ is not unique with this property.

**For example,**
As explained in Theorem 2.19, consider the geodesic $P$: $v_2v_1v_3$ in Figure 2.5.



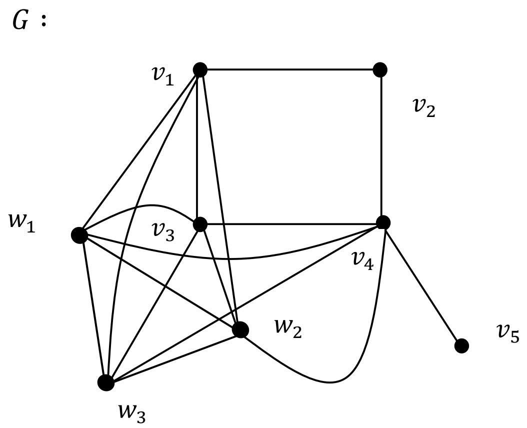



**Figure 2.5**

Here $S_1 = \{v_2\}$ is a minimum $\partial_R - set$ of $H$, $\partial_R(H) = 1$ and $\partial_R(G_1) = 4$.

**Theorem 2.20**
For any given natural number $m$, there exists a graph $G$ such that $\partial_R(G) = |Cen(G)| = m$.
***Proof*** Let $m$ be any natural number.
Construct a graph $G$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5, u_1, u_2, \dots, u_m\}$ and $E(G) = \{v_iv_{i+1}, v_2u_j, v_3u_j, v_4u_j, v_4u_m$ / $1 \leq i \leq 4$ $and$ $1 \leq j \leq m-1\}$. We calim that $\partial_R(G) = |Cen(G)|$. Since the radius of $G$ is 2. Then $Cen(G) = \{v_3, u_1, u_2, \dots, u_{m-1}\}$
That is, $|Cen(G)| = m$. ……(1)
Let $S = \{u_m\} \subseteq V(G)$. Then $B_R(S) = \{v_3, v_5, u_1, u_2, \dots, u_{m-1}\}$. Therefore, the max-radial number of the set $S$ is $\partial_R(S) = |B_R(S)| - |S| = m$. Suppose any set $S$ contains atleast two vertices. Then $B_R(S)$ contains less than or equal to $m+2$ vertices. Therefore, $\partial_R(S) = |B_R(S)| - |S| \leq m$. Thus $\partial_R(G) = m$. ……(2)
From (1) and (2), $\partial_R(G) = |Cen(G)| = m$.
This complete the proof. □
**For example,**

when $m = 3$ in Theorem 2.20, the constructed graph $G$ is shown in Figure 2.6.

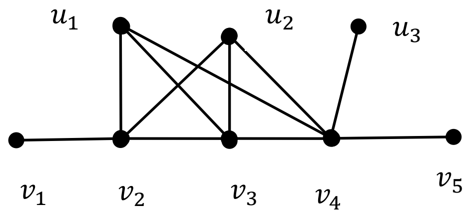



**Figure 2.6**

Here $S = \{v_2\}$ is a minimum $\partial_R - set$ of $G$, $\partial_R(G) = 3$ and $|Cen(G)| = 3$.

**Theorem 2.21**
For any positive integer $m$, there exists a graph $G$ such that $\partial_R(G) = \chi(G) = m$.
***Proof***

Let $m$ be any positive integer. Construct a graph $G$ with $V(G) = \{v_1, v_2, v_3, v_4, u_1, u_2, ...,$
$u_{m-2}\}$ and $E(G) = \{v_1v_2, v_1v_3, v_1v_4, u_iu_j,$
$v_1u_i, v_2u_i$ / $1 \leq i, j \leq m-2\}$. We claim that $\partial_R(G) = \chi(G) = m$. Let $v_1$ be a full vertex of $G$. Then quote by Fact Theorem 2.16, $\partial_R(G) = (m+2) - 2 = m$.
........(1)
Suppose $\{x_1, x_2, \dots, x_m\}$ is the set of colours. Then $\{v_1\}, \{v_2, v_3, v_4\}, \{u_1\}, \{u_2\}, \dots, \{u_{m-2}\}$ are the colour classes induced by $x_1, x_2, \dots, x_m$ respectively. Therefore, $m$-colouring of $G$ exists. Hence the Chromatic number $\chi(G) = m$.
.........(2)
From (1) and (2), $\partial_R(G) = \chi(G)$. □
**For example,**
when $m = 4$ in Theorem 2.21, the constructed graph $G$ is shown in Figure 2.7.

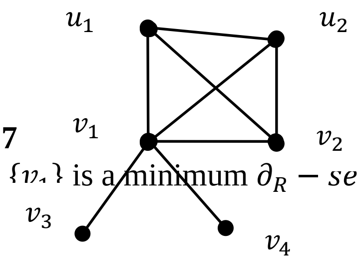



**Figure 2.7**
Here $S = \{v_1\}$ is a minimum $\partial_R - set$ of $G$, $\partial_R(G) = 4$ and $\chi(G) = 4$ .

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