International Journal of Aquatic Science
ISSN: 2008-8019
Vol 12, Issue 02, 2021

Parameter-Uniform Convergence Of A Finite Element Method For A Partially Singularly Perturbed Linear Reaction-Diffusion System

M. Vinoth[1], M. Joseph Paramasivam[2]

[1,2]Department of Mathematics, Bishop Heber college, Tiruchirappalli, Tamilnadu, India*
Affiliated to Bharathidasan University

Email: vinothbhcedu@gmail.com[1], sivambhcedu@gmail.com[2]

Abstract: The reaction-diffusion kind of a partially singularly perturbed linear system of 'n' second order ordinary differential equations is considered. The leading terms first 'm' equation's are multiplied by a small positive parameters and the remaining '$n-m$' equations are not singularly perturbed. It is assumed that these 'm' singular perturbation parameters are distinct. First 'm' solution's elements have overlapping boundary layers and remaining 'n-m' solution's elements have less serve overlapping layers. On a piecewise uniform Shishkin mesh, a numerical system is built that employs the finite element method. The numerical approximations obtained by this approach are proven to be effectively almost second order convergent uniformly with respect to all perturbation parameters. In support of the theory, numerical illustrations are given.

Keywords: Partially singularly perturbed problems, reaction - diffusion equations, overlapping boundary layers, finite element method, Shishkin mesh and parameter - uniform convergence.

1. INTRODUCTION

In the interval $\Omega = \{x: 0 < x < 1\}$, a partially singularly perturbed linear system of 'n' second order ordinary differential equations of reaction - diffusion form is considered.
The self-adjoint two-point boundary value problem that corresponds is

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \quad on\,\Omega, \vec{u} \;\; given\, on \;\; \Gamma \tag{1.1}$$

where $\Gamma = \{0,1\}$ $\;\overline{\Omega} = \Omega \cup \Gamma$.
Here $\vec{u}$ is a column n-vector, E and $A(x)$ are $n \times n$ matrices, $E = diag(\vec{\varepsilon}), \vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1, \dots, n$.
The parameters $\varepsilon_i,\ i = 1, \dots, m$ are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_m < \varepsilon_{m+1} = \cdots = \varepsilon_n = 1.$$

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}(x)| \;\; for\; 1 \leq i \leq n \; and \;\; a_{ij}(x) \leq 0 \;\; for \;\; i \neq j \tag{1.2}$$

and, for some α, $0 < \alpha < \min_{\substack{x\in[0,1]\\ 1\le i\le n}} \sum_{j=1}^{n} |a_{ij}(x)|$. (1.3)

It is assumed that $a_{ij}, f_i \in C^{(2)}(\overline{\Omega})$, for $i, j = 1, \ldots, n$. Then (1.1) has a solution $\vec{u} \in C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega)$.

It is also assumed that

$$\sqrt{\varepsilon_m} \le \frac{\sqrt{\alpha}}{6}. \tag{1.4}$$

C is a generalised positive constant that is independent of x as well as m singular perturbation and discretization parameters used in this article. This is the outline for the document. The boundary layers are resolved using piecewise-uniform Shishkin meshes in Section 3. In Section 4 the discrete problem is defied and the corresponding maximum principle and stability result are established. The parameter-uniform error estimation is described and illustrated in Section 6. The numerical diagrams in Section 8 are included.

2. ANALYSIS OF THE FINITE ELEMENT METHOD

Let V be a given Hilbert space with norm $\|.\|_V$ and scalar product $(\cdot,\cdot)$. V is usually a subspace of the Sobolev space $H^1(\Omega)$.

Consider the weak formulation, find $\vec{u} \in H_0^1(0,1)^n$ in particular $u_i \in H_0^1(0,1)$ for $i = 1, \ldots, n$ such that

$$\beta_i(u_i, v_i) = f_i(v_i) \ \forall \ v_i \in H_0^1(0,1). \tag{3.1}$$

For $i = 1, \ldots, m$,

$$\beta_i(u_i, v_i) = -\varepsilon_i(u_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}u_j), v_i\right)$$

and

$$f_i(v_i) = (f_i, v_i).$$

For $i = m+1, \ldots, n$,

$$\beta_i(u_i, v_i) = -(u_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}u_j), v_i\right)$$

and

$$f_i(v_i) = (f_i, v_i).$$

where $(u_i, v_i) = \int_0^1 u_i v_i \, dx$. $\beta_i(u_i, v_i)$ is a bilinear form on $H_0^1(0,1)^n$ and $f_i(v_i)$, a given continuous linear functional on $H_0^1(0,1)^n$.

Lemma 3.1 *Suppose that the bilinear form $\beta_i(\cdot,\cdot)$, $i = 1, \ldots, n$, is continuous on $H_0^1(0,1)^n$ is coercive, that*

$$|\beta_i(u_i, v_i)| \le \gamma \parallel u_i \parallel . \parallel v_i \parallel \tag{3.2}$$

$$\beta_i(v_i, v_i) \ge \alpha \parallel v_i \parallel^2 \tag{3.3}$$

where α and γ are constants that are indepentent of u_i and v_i. Then for any continuous linear functional $f_i(\cdot)$, the problem (3.1) has a unique solution.

A natural norm on $H_0^1(0,1)^n$ associated with the bilinear form $\beta_i(\cdot,\cdot)$ is the energy norm

$$\parallel v_i \parallel_{\varepsilon_i}^2 = (\varepsilon_i \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2)$$

where $\parallel v_i \parallel_1 = (v_i', v_i')^{\frac{1}{2}}$, $\parallel v_i \parallel_0 = (v_i, v_i)^{\frac{1}{2}}$ on $H_0^1(0,1)^n$.

Lemma 3.2 *A bilinear functional $\beta_i(u_i, v_i), i = 1, \dots, n$, satisfies the coercive property with respect to*

$$\parallel v_i \parallel_{\varepsilon_i}^2 \leq \beta_i(v_i, v_i)$$

Proof: For $i = 1, \dots, m$

$$\beta_i(v_i, v_i) = -\varepsilon_i(v_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}v_j), v_i\right)$$

$$= \varepsilon_i \parallel v_i \parallel_1^2 + \int_0^1 \left(\sum_{j=1}^{n}(a_{ij}v_j) \cdot v_i\right) dx$$

$$\geq \varepsilon_i \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2.$$

For $i = m + 1, \dots, n$

$$\beta_i(v_i, v_i) = -(v_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}v_j), v_i\right)$$

$$= \parallel v_i \parallel_1^2 + \int_0^1 \left(\sum_{j=1}^{n}(a_{ij}v_j) \cdot v_i\right) dx$$

$$\geq \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2.$$

1. **The Shishkin mesh**

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed. Let $\Omega^N = \{x_k\}_{k=1}^{N-1}$, $\overline{\Omega}^N = \{x_k\}_{k=0}^{N}$ and $\Gamma^N = \Gamma$. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on $[0,1]$ that was generated by dividing $[0,1]$ into $2m + 1$ mesh-intervals as follows:

$$[0, \sigma_1] \cup \dots \cup (\sigma_{m-1}, \sigma_m] \cup (\sigma_m, 1 - \sigma_m] \cup (1 - \sigma_m, 1 - \sigma_{m-1}] \cup \dots \cup (1 - \sigma_1, 1].$$

The points separating the uniform meshes are determined by the m parameters σ_r which are defined by $\sigma_0 = 0, \sigma_{m+1} = \frac{1}{2}$,

$$\sigma_m = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_m}}{\sqrt{\alpha}} \ln N\right\} \tag{4.1}$$

and, for $r = m - 1, \dots, 1$,

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, \frac{2\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{4.2}$$

Clearly

$$0 < \sigma_1 < \dots < \sigma_m \leq \frac{1}{4}, \qquad \frac{3}{4} \leq 1 - \sigma_m < \dots < 1 - \sigma_1 < 1.$$

Then a uniform mesh of $\frac{N}{2}$ mesh-points is placed on the sub-interval $(\sigma_m, 1 - \sigma_m]$, and a uniform mesh of $\frac{N}{4m}$ mesh-points is placed on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(1 - \sigma_{r+1}, 1 - \sigma_r]$, $r = 0, 1, \dots, m - 1$, respectively. In practice, it is convenient to take $N = 4\,m\,\delta$, $\delta \geq 3$, where m denotes the number of distinct singular perturbation parameters involved in the experiment (1.1). This produces a class of 2^m piecewise uniform Shishkin meshes $\overline{\Omega}^N$. When all of the parameters $\sigma_r = \frac{r}{4N}$, $r = 1, \dots, m$, are set to the left, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh with the transformation parameters σ_r and a scale N^{-1} from 0 to 1. The following inequalities hold for the mesh Ω^N, $s = 1, \dots, m-1$

$$\begin{aligned} & h_k \le \frac{2}{N} \quad for \quad 1 \le k \le N \\ & h_k \ge \frac{1}{N} \quad for \quad \frac{N}{4} \le k \le \frac{3N}{4} \\ & h_k \le \frac{1}{N} \quad for \quad 1 \le k \le \frac{N}{4} \ and \ \frac{3N}{4} \le k \le N \\ & h_k \ge \frac{N}{4s} \quad for \quad \frac{N}{4s} \le k \le \frac{N}{4(s+1)} \ and \ \left(1 - \frac{N}{4(s+1)}\right) \le k \le \left(1 - \frac{N}{4s}\right) \\ & h_k \le \frac{N}{4s} \quad for \quad 1 \le k \le \frac{N}{4(s)} \ and \ \left(1 - \frac{N}{4(s)}\right) \le k \le N. \end{aligned} \tag{4.4}$$

2. **The discrete problem**

In this segment, a numerical method for (3.1) is constructed using a finite element method with a suitable Shishkin mesh. Let for $i = 1,2, \dots, n$ and $k = 1,2, \dots, N-1, V_{i,k} \subset H_0^1(0,1)^n$ be the space of piecewise linear functionals on Ω, that vanish at $x = 0$ and 1.

The finite element approach is now established for the discrete two-point boundary value problem, $U_{i,k} \in V_{i,k}$

$$\beta_i(U_{i,k}, v_{i,k}) = f(v_{i,k}) \qquad \forall \quad v_{i,k} \in V_{i,k} \tag{5.1}$$

By Lax-Migram, Lemma implies that

1. The discrete problem has a unique solution,
2. The discrete problem is stable.

From (1.3) on A implies that for arbitrary $x \in (0,1)$

$$\xi^T A \xi \ge \alpha \xi^T \xi \quad \forall \quad \xi \ on \ V_{i,k}^*$$

where $V_{i,k}^*$ is dual space for $V_{i,k}$.

Let $\{\phi_{i,k}: k = 1, \cdots, N-1\}$ be a basis for $V_{i,k}$, where $N = N(i,k)$ is the dimension of $V_{i,k}$. Then

$$U_{i,k} = \sum_{k=1}^{N-1} C_{i,k} \phi_{i,k}$$

where the unknowns $C_{i,k}$ satisfiy the linear system

$$AU = B$$

with $A = \beta_i(\phi_{i,k_1}, \phi_{i,k_2}), U = C_{i,k}, B = f_i(\phi_{i,k})$.

The corresponding difference scheme is

$$\begin{pmatrix} \beta_1(\phi_{1,1}, \phi_{1,1}) & \beta_1(\phi_{1,1}, \phi_{1,2}) & \cdots & \beta_i(\phi_{1,1}, \phi_{n,N-1}) \\ \beta_1(\phi_{1,2}, \phi_{1,1}) & \beta_1(\phi_{1,2}, \phi_{1,2}) & \cdots & \beta_1(\phi_{1,2}, \phi_{n,N-1}) \\ \vdots & \vdots & & \vdots \\ \beta_n(\phi_{n,N-1}, \phi_{n,1}) & \beta_n(\phi_{n,N-1}, \phi_{n,2}) & \cdots & \beta_n(\phi_{n,N-1}, \phi_{n,N-1}) \end{pmatrix} \begin{pmatrix} C_{1,1} \\ C_{1,2} \\ \vdots \\ C_{n,N-1} \end{pmatrix} = \begin{pmatrix} (f_1, \phi_{1,1}) \\ (f_1, \phi_{1,2}) \\ \vdots \\ (f_n, \phi_{n,N-1}) \end{pmatrix}.$$

$For\ k = 1, \dots, N-1$

$$\phi_{1,k} = \phi_{2,k} = \cdots = \phi_{n,k}$$
$$C_{1,k} = C_{2,k} = \cdots = C_{n,k}.$$

The nonzero contribution from a particular element is

$$A_{i,k} = \begin{pmatrix} \int_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k-1}dx & \int_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k}dx \\ \int_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k}dx & \int_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k+1}dx \end{pmatrix}.$$

Similarly, the local load vector is

$$B_{i,k} = \begin{pmatrix} \int_{x_k}^{x_{k+1}} f_i.\phi_{i,k}dx \\ \int_{x_k}^{x_{k+1}} f_i.\phi_{i,k+1}dx \end{pmatrix}.$$

3. Interpolation error bounds

Lemma 6.1. *Let* $u_{i,k}^*$ *be the* $V_{i,k}$*-interpolant of the solution* $u_{i,k}$ *of* (1.1) *on the fitted mesh* Ω^N*. Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} \| u_{i,k}^* - u_{i,k} \|_{\Omega^N} \leq C\left(N^{-1}lnN\right)^2,$$

where C *is a constant independent of the parameters* ε_i*.*

Proof: The estimate is obtained separately on each subinterval $\Omega_k = (x_{k-1}, x_k)$, $k = 1, \dots, N-1$. Note that for any function $g_{i,k}$ on Ω_k

$$g_{i,k}^* = g_{i,k-1}\phi_{i,k-1} + g_{i,k}\phi_{i,k},$$

and so it is obvious that, on Ω_k,

$$\left|g_{i,k}^*(x)\right| \leq \max_{\Omega_k}\left|g_{i,k}(x)\right|, \tag{6.1}$$

and it's easy to see that by using sufficient Taylor expansions

$$\left|g_{i,k}^*(x) - g_{i,k}(x)\right| \leq Ch_k^2 \max_{\Omega_k}\left|g_{i,k}''(x)\right|. \tag{6.2}$$

From (6.2) and Lemma 2.3, on Ω_k,

$$\begin{aligned}\left|u_{i,k}^*(x) - u_{i,k}(x)\right| &\leq Ch_k^2 \max_{\Omega_k}\left|u_{i,k}''(x)\right| \\ &\leq C\frac{h_k^2}{\varepsilon_i}. \end{aligned}\tag{6.3}$$

Also, using Lemma 2.3, Lemma 2.4 and Lemma 2.5 on Ω_k,

$$\begin{aligned}&\left|u_{i,k}^*(x) - u_{i,k}(x)\right| \\ &= \left|v_{i,k}^*(x) + w_{i,k}^*(x) - v_{i,k}(x) - w_{i,k}(x)\right| \\ &\leq \left|v_{i,k}^*(x) - v_{i,k}(x)\right| + \left|w_{i,k}^{*L}(x) - w_{i,k}^L(x)\right| + \left|w_{i,k}^{*R}(x) - w_{i,k}^R(x)\right| \\ &\leq Ch_k^2 \max_{\Omega_k}\left|v_{i,k}''(x)\right| + Ch_k^2 \max_{\Omega_k}\left|w^{L_{i,k}''(x)}\right| + Ch_k^2 \max_{\Omega_k}\left|w_{i,k}^{R''}(x)\right|\end{aligned}$$

For $i = 1, \dots, m$

$$\leq C\left(\left(1 + \sum_{q=i}^{m} B_q(x)\right) + \sum_{q=i}^{m}\frac{B_q^L(x)}{\varepsilon_q} + \sum_{q=i}^{m}\frac{B_q^R(x)}{\varepsilon_q}\right) \tag{6.4}$$

The discussion now centres on whether $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \geq \frac{1}{4}$ or $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \leq \frac{1}{4}$ should be used. In the first case $\frac{1}{\varepsilon_m} \leq C(\ln N)^2$ and the result follows at once from (4.4) and (6.3). In the second case $\sigma_m = \frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}}$. Suppose that k satisfies $\frac{N}{4} \leq k \leq \frac{3N}{4}$. Then $h_k = \frac{2(1-2\sigma_m)}{N}$ and therefore

$$\frac{h_k}{\varepsilon_m} = 2N^{-1}\frac{1-2\sigma_m}{\varepsilon_m},$$

$\sigma_m \leq 1 - x_k$, and so

$$e^{-\frac{\sqrt{\alpha}(1-x_k)}{\sqrt{\varepsilon_m}}} \leq e^{-\frac{\sqrt{\alpha}\sigma_m}{\sqrt{\varepsilon_m}}} = e^{-2\ln N} = N^{-2}. \quad (6.5)$$

Using (6.5) and (4.4) in (6.4) gives the required result.

On the other hand, if k satisfies $1 \leq k \leq \frac{N}{4}$ and $\frac{3N}{4} \leq k \leq N$ and $r = m-1, \dots, 1$, then the discussion now centres on whether $2\frac{\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \geq \frac{r\sigma_{r+1}}{r+1}$ or $2\frac{\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \leq \frac{r\sigma_{r+1}}{r+1}$ should be used. In the first case $\frac{1}{\varepsilon_r} \leq C(\ln N)^2$ and the result follows at once from (4.4) and (6.3).

In the second case $\sigma_r = 2\sqrt{\varepsilon_r}\ln N/\sqrt{\alpha}$ and for $s = 1, \dots, m-1$,

1. suppose that k satisfies $\frac{N}{4(s+1)} \leq k \leq \frac{N}{4(s)}$ and $1 - \left(\frac{N}{4(s)}\right) \leq k \leq 1 - \left(\frac{N}{4(s+1)}\right)$. Then $h_k = \frac{4m(\sigma_{r+1}-\sigma_r)}{N}$ *or* $\frac{4m(\sigma_r-\sigma_{r+1})}{N}$ and $\sigma_r \leq 1 - x_k$ therefore
$$\frac{h_k}{\sqrt{\varepsilon_r}} = 4mN^{-1}\frac{\sigma_{r+1}-\sigma_r}{\varepsilon_r} \quad or \quad 4mN^{-1}\frac{\sigma_r-\sigma_{r+1}}{\varepsilon_r}. \quad (6.6)$$
Using (6.6) and (4.4) in (6.4) gives the required result.
2. If k satisfies $1 \leq k \leq \frac{N}{4(s+1)}$ and $1 - \left(\frac{N}{4(s+1)}\right) \leq k \leq N$, then $h_k = 4m\frac{(\sigma_{r+1}-\sigma_r)}{N}$ or $4m\frac{(\sigma_{r+1}-\sigma_r)}{N}$, and therefore
$$\frac{h_k}{\varepsilon_r} = 4mN^{-1}\frac{(\sigma_{r+1}-\sigma_r)}{\varepsilon_r} \quad or \quad 4mN^{-1}\frac{(\sigma_r-\sigma_{r+1})}{\varepsilon_r}, \quad (6.7)$$
Using (6.7) and (4.4) in (6.4) gives the required result.

For $i = m+1, \dots, n$

$$\leq C\left(1 + B_m(x) + C_1 B_m^L(x) + C_2\varepsilon_m\left(1 - B_m^L(x)\right) + C_1 B_m^R(x) + C_2\varepsilon_m\left(1 - B_m^R(x)\right)\right)$$

This gives the required result.

Lemma 6.2. *Let $u_{i,k}^*$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of* (1.1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n}\ \sup_{0<\varepsilon_i\leq 1} \| u_{i,k}^* - u_{i,k} \|_{\varepsilon_i} \leq C\left(N^{-1}\ln N\right)^2,$$

where C is a constant independent of ε_i.

Proof:

For $i = 1, \dots, m$ from the definition of the energy norm

$$\| u_{i,k}^* - u_{i,k} \|_{\varepsilon_i}^2 = \varepsilon_i\left(\left(u_{i,k}^* - u_{i,k}\right)', \left(u_{i,k}^* - u_{i,k}\right)'\right) + \alpha\left(u_{i,k}^* - u_{i,k}, u_{i,k}^* - u_{i,k}\right). \quad (6.8)$$

Each term on the R.H.S of (6.8) is now treated separately. It is easy to see that the second term satisfies

$$\left(u_{i,k}^* - u_{i,k}, u_{i,k}^* - u_{i,k}\right) \leq \| u_{i,k}^* - u_{i,k} \|^2. \quad (6.9)$$

Using integration by parts and noting that $\left(u_{i,k}^* - u_{i,k}\right)(x_k) = 0$, for each k, the first term can be bounded as follows

$$\varepsilon_i\left(\left(u_{i,k}^*-u_{i,k}\right)',\left(u_{i,k}^*-u_{i,k}\right)'\right)$$
$$=\varepsilon_i\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}\left(u_{i,k}^{*\,'}(s)-u_{i,k}'(s)\right)^2 ds$$
$$=-\varepsilon_i\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}\left(u_{i,k}^{*\,''}(s)-u_{i,k}''(s)\right)(u_{i,k}^*(s)-u_{i,k}(s))ds$$
$$=\varepsilon_i\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}u_{i,k}''(s)\left(u_{i,k}^*(s)-u_{i,k}(s)\right)ds$$
$$=\left(\varepsilon_i\, u_{i,k}'',u_{i,k}^*-u_{i,k}\right),$$

where the fact that $u_{i,k}^{*''}=0$ on each Ω_k has been used.
The estimate for the second derivative of the components of $u_{i,k}$ in lemma 2.4 and lemma 2.5 then gives

$$\left|\left(\varepsilon_i\, u_{i,k}'',u_{i,k}^*-u_{i,k}\right)\right|\le\| u_{i,k}^*-u_{i,k}$$
$$\|\int_0^1\varepsilon_i\left|u_{i,k}''\right|ds$$
$$\left|\left(\varepsilon_i\, u_{i,k}'',u_{i,k}^*-u_{i,k}\right)\right|\le\| u_{i,k}^*-u_{i,k}$$
$$\|\int_0^1\left(\varepsilon_i\left|v_{i,k}''\right|+\varepsilon_i\left|w_{i,k}^{L,''}\right|+\varepsilon_i\left|w_{i,k}^{R,''}\right|\right)ds$$
$$\le C\,\| u_{i,k}^*-u_{i,k}\|\int_0^1\left(\varepsilon_i\left(1+\sum_{q=i}^{n}B_q(s)\right)+\varepsilon_i\sum_{q=i}^{n}\frac{B_q^L(s)}{\varepsilon_q}+\varepsilon_i\sum_{q=i}^{n}\frac{B_q^R(s)}{\varepsilon_q}\right)ds$$
$$\le\ C\,\| u_{i,k}^*-u_{i,k}\|,$$

and so

$$\varepsilon_i\left(\left(u_{i,k}^*-u_{i,k}\right)',\left(u_{i,k}^*-u_{i,k}\right)'\right)\le\ C\,\| u_{i,k}^*-u_{i,k}\|. \qquad (6.10)$$

Combining (6.8) − (6.10) leads to

$$\| u_{i,k}^*-u_{i,k}\|_{\varepsilon_i}^2\le\ C\,\| u_{i,k}^*-u_{i,k}\|\left(1\ +\alpha\,\| u_{i,k}^*-u_{i,k}\|\right)$$

For $i=m+1,\ldots,n$ from the definition of the energy norm

$$\| u_{i,k}^*-u_{i,k}\|_{\varepsilon_i}^2\ =\left(\left(u_{i,k}^*-u_{i,k}\right)',\left(u_{i,k}^*-u_{i,k}\right)'\right)+\alpha\left(u_{i,k}^*-u_{i,k},u_{i,k}^*-u_{i,k}\right). \quad (6.11)$$

Each term on the R.H.S of (6.11) is now treated separately. It is easy to see that the second term satisfies

$$\left(u_{i,k}^*-u_{i,k},u_{i,k}^*-u_{i,k}\right)\le\| u_{i,k}^*-u_{i,k}\|^2. \qquad (6.12)$$

Using integration by parts and noting that $\left(u_{i,k}^*-u_{i,k}\right)(x_k)=0$, for each k, the first term can be bounded as follows

$$\left(\left(u_{i,k}^*-u_{i,k}\right)',\left(u_{i,k}^*-u_{i,k}\right)'\right)$$
$$=\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}\left(u_{i,k}^{*\,'}(s)-u_{i,k}'(s)\right)^2 ds$$
$$=-\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}\left(u_{i,k}^{*\,''}(s)-u_{i,k}''(s)\right)(u_{i,k}^*(s)-u_{i,k}(s))ds$$

$$= \sum_{k=1}^{N-1} \int_{x_{k-1}}^{x_k} u''_{i,k}(s)\left(u^*_{i,k}(s) - u_{i,k}(s)\right) ds$$
$$= \left(u''_{i,k}, u^*_{i,k} - u_{i,k}\right),$$

where the fact that $u^{*''}_{i,k} = 0$ on each Ω_k has been used.
The estimate for the second derivative of the components of $u_{i,k}$ in lemma 2.4 and lemma 2.5 then gives

$$\left|\left(u''_{i,k}, u^*_{i,k} - u_{i,k}\right)\right| \leq \| u^*_{i,k} - u_{i,k} \| \int_0^1 \left|u''_{i,k}\right| ds$$
$$\left|\left(u''_{i,k}, u^*_{i,k} - u_{i,k}\right)\right| \leq \| u^*_{i,k} - u_{i,k} \| \int_0^1 \left(\left|v''_{i,k}\right| + \left|w^{L,''}_{i,k}\right| + \left|w^{R,''}_{i,k}\right|\right) ds$$
$$\leq C \| u^*_{i,k} - u_{i,k} \|$$
$$\int_0^1 \left(1 + B_m(s) + C_1 B^L_m(s) + C_2\left(1 - B^L_m(s)\right) + C_1 B^L_m(s) + C_2\left(1 - B^L_m(s)\right)\right) ds$$
$$\leq C \| u^*_{i,k} - u_{i,k} \|,$$

and so

$$\left(\left(u^*_{i,k} - u_{i,k}\right)', \left(u^*_{i,k} - u_{i,k}\right)'\right) \leq C \| u^*_{i,k} - u_{i,k} \|. \qquad (6.13)$$

Combining $(6.11) - (6.13)$ leads to

$$\| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i} \leq C \| u^*_{i,k} - u_{i,k} \| \left(1 + \alpha \| u^*_{i,k} - u_{i,k} \|\right)$$

and the proof is completed using the estimate of $\| u^*_{i,k} - u_{i,k} \|$ from Lemma 6.1.

Lemma 6.3. *Let* $u^*_{i,k}$ *be the* $V_{i,k}$ *-interpolant of the solution* $u_{i,k}$ *of* (1.1) *on the fitted mesh* Ω^N*. Then*

$$\max_{i=1,\dots,n} \| u^*_{i,k} - u_{i,k} \|_{\varepsilon_i, \overline{\Omega}^N} \leq C\left(N^{-1} \ln N\right)^2.$$

Proof: Since $u^*_{i,k}(x_k) - u_{i,k}(x_k) = 0$, it follows from the definitions of the norms that

$$\| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i, \overline{\Omega}^N} = \varepsilon_i \left(\left(u^*_{i,k} - u_{i,k}\right)', \left(u^*_{i,k} - u_{i,k}\right)'\right) \leq \| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i}.$$

Using the estimate in Lemma 6.2 completes the proof.

4. **Interpolation error estimate**

Lemma 7.1. *Let* $u_{i,k}$ *be the solution of (1.1) and* $U_{i,k}$ *the solution of (5.1). Suppose that* $v_i \in V_{i,k}$*. Then*

$$\max_{i=1,\dots,n} \left|\beta_i\left(U_{i,k} - u_{i,k}, v_i\right)\right| \leq C\left(N^{-1} \ln N\right) \| v_i \|_{l^2\left(\overline{\Omega}^N\right)},$$

where the constant C *is independent of* ε_i*.*

Proof: Since v_i is in $V_{i,k}$, it can be written in the form

$$v_i = \sum_{k=1}^{N-1} v_{i,k} \phi_{i,k},$$

and so

$$\beta_i\left(U_{i,k} - u_{i,k}, v_i\right) = \sum_{k=1}^{N-1} v_{i,k} \beta_i\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right). \qquad (7.1)$$

Then, for each k, $1 \leq k \leq N-1$, using (1.1), (5.1) and the fact that $\left(1, \phi_{i,k}\right)_{\Omega^N} = \left(1, \phi_{i,k}\right) = \frac{h_k + h_{k+1}}{2}$,

$$\beta_i\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right) = \sum_{j=1}^{n}\left(a_{ij}U_{i,k}, \phi_{i,k}\right) - \sum_{j=1}^{n}\left(a_{ij}u_{i,k}, \phi_{i,k}\right)$$

$$= \sum_{j=1}^{n}\left(a_{ij}u_{j,k}(x_k), \phi_{i,k}\right) - \sum_{j=1}^{n}\left(a_{ij}u_{j,k}, \phi_{i,k}\right).$$

$$= \sum_{j=1}^{n}\left(a_{ij}\left(u_{j,k}(x_k) - u_{j,k}\right), \phi_{i,k}\right)$$

Since

$$\left|u_{j,k}(x_k) - u_{j,k}\right| = \left|\int_{x}^{x_k} u_{j,k}^{'}(s)ds\right| \leq I_k,$$

where

$$I_k = \int_{x_{k-1}}^{x_{k+1}} \left|u_{j,k}^{'}(s)\right|ds,$$

it follows from (4.4) that

$$\beta_i\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right)| \leq C\frac{(h_k + h_{k+1})}{2}\left(I_k + N^{-1}\right). \tag{7.2}$$

Assume for the moment that

$$I_k \leq CN^{-1}\ln N. \tag{7.3}$$

Then (7.1)-(7.3) and the Cauchy-Schwarz inequality give

$$\left|\beta_i\left(U_{i,k} - u_{i,k}, v_i\right)\right| \leq CN^{-1}\ln N \sum_{k=1}^{N-1}\frac{(h_k + h_{k+1})^{\frac{1}{2}}}{2}\left|v_{i,k}\right|\frac{(h_k + h_{k+1})^{\frac{1}{2}}}{2}$$

$$\leq CN^{-1}\ln N \parallel v_{i,k} \parallel_{l^2(\overline{\Omega}^N)},$$

as required.

It remains therefore to verify (7.3). From the estimate in Lemma 2.3 for the first derivative of the solution, it is clear that

$$I_k \leq C\int_{x_{k-1}}^{x_{k+1}} \varepsilon_i^{-\frac{1}{2}}\left(\parallel \vec{u} \parallel_{\Gamma} + \parallel \vec{f} \parallel_{\Omega}\right)dx.$$

It follows that

$$I_k \leq C\frac{(h_k + h_{k+1})}{2}/\sqrt{\varepsilon_i}, \tag{7.4}$$

and that $i = 1, \ldots, m$

$$I_k \leq C\frac{h_k + h_{k+1}}{2} + e^{-\frac{\sqrt{\alpha}(1-x_{k+1})}{\sqrt{\varepsilon_m}}}, \tag{7.5}$$

The argument now depends on whether $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \geq \frac{1}{4}$ or $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \leq \frac{1}{4}$. In the first case $\frac{1}{\sqrt{\varepsilon_m}} \leq C\ln N$ and the result follows at once from (4.4) and (7.4). In the second case $\sigma_m = \frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}}$. Suppose that k satisfies $\frac{N}{4} < k < \frac{3N}{4}$. Then $h_k = \frac{2(1-2\sigma_m)}{N}$ and therefore

$$\frac{h_k}{\sqrt{\mathring{a}_m}} = 2N^{-1}\frac{(1-2\sigma_m)}{\sqrt{\varepsilon_m}},$$

$\sigma_m \le 1 - x_{k+1}$, and so

$$e^{-\frac{\sqrt{\alpha}(1-x_{k+1})}{\sqrt{\varepsilon_m}}} \le e^{-\frac{\sqrt{\alpha}\sigma_m}{\sqrt{\varepsilon_m}}} = e^{-2\ln N} = N^{-2}. \qquad (7.6)$$

Using (7.6), (4.4) and (7.4) gives the required result.

On the other hand, if k satisfies $1 \le k < \frac{N}{4}$ and $\frac{3N}{4} < k \le N$ and $r = m-1, \dots, 1$ then the argument now depends on whether $\frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \ge \frac{r\sigma_{r+1}}{r+1}$ or $\frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\mathring{a}}} \le \frac{r\sigma_{r+1}}{r+1}$, In the first case $\frac{1}{\sqrt{\varepsilon_r}} \le C\ln N$ and the result follows at once from (4.4) and (7.4). In the second case $\sigma_r = \frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}}$ and for $s = 1, \dots, m-1$.

(1) Suppose that k satisfies $\frac{N}{4(s+1)} < k < \frac{N}{4(s)}$ and $1 - \left(\frac{N}{4(s)}\right) < k < 1 - \left(\frac{N}{4(s+1)}\right)$. Then $h_k = 4m\frac{(\sigma_{r+1}-\sigma_r)}{N}$ *or* $4m\frac{(\sigma_r-\sigma_{r+1})}{N}$ and $\sigma_r \le 1 - x_k$ therefore

$$\frac{h_k}{\sqrt{\varepsilon_r}} = 4mN^{-1}\frac{\sigma_{r+1}-\sigma_r}{\sqrt{\varepsilon_r}} \quad or \quad 4mN^{-1}\frac{\sigma_r-\sigma_{r+1}}{\sqrt{\varepsilon_r}}. \qquad (7.7)$$

Using (7.7) and (4.4) in (7.4) gives the required result.

(2) If k satisfies $1 \le k < \frac{N}{4(s+1)}$ and $1 - \left(\frac{N}{4(s+1)}\right) < k < N$, then $h_k = \frac{4m(\sigma_{r+1}-\sigma_r)}{N}$ or $4m\frac{(\sigma_{r+1}-\sigma_r)}{N}$ and therefore

$$\frac{h_k}{\sqrt{\varepsilon_r}} = 4mN^{-1}\frac{(\sigma_{r+1}-\sigma_r)}{\sqrt{\varepsilon_r}} \; or \; 4mN^{-1}\frac{\sigma_r-\sigma_{r+1}}{\sqrt{\varepsilon_r}}, \qquad (7.8)$$

Using (7.8) and (4.4) in (7.4) gives the required result.

(3) Finally, suppose that $k = \left\{\frac{N}{4(s)}, 1-\left(\frac{N}{4(s)}\right), \frac{N}{4m}, 1-\left(\frac{N}{4m}\right)\right\}$. Then

$$I_k \le \left(\int_{k-1}^{k} + \int_{k}^{k+1}\right)|u'_{i,k}|dx < I_{k-1} + I_{k+1}$$
$$\le CN^{-1}\ln N$$

For $i = m+1, \dots, n$, using (4.4) in (7.4) gives the required.

5. Discretization error

Lemma 8.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of (1.1) and $U_{i,k}$ the solution of (5.1). Then*

$$\max_{i=1,\dots,n} \| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \le C\left(N^{-1}\ln N\right)^2,$$

where the constant C is independent of the parameters ε_i.

Proof: From the coercivity of $\beta_i(,)$ in Lemma 3.1 and since $U_{i,k} - u^*_{i.k} \in V_{i,k}$

$$\| U_{i,k} - u^*_{i,k} \|^2_{\varepsilon_i,\Omega^N} \le C\,\beta_i(U_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})$$
$$\le C[\beta_i(U_{i,k} - u_{i,k}, U_{i,k} - u^*_{i,k}) + \beta_i(u_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})]$$

Using Lemma 7.1, with $v_i = U_{i,k} - u^*_{i,k}$, then gives

$$\| U_{i,j} - u^*_{i,k} \|^2_{\varepsilon_i,\overline{\Omega}^N} \le C\,(N^{-1}\ln N)^2 \| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N}.$$

Cancelling the common factor gives

$$\| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \le C(N^{-1}\ln N)^2,$$

as required.

Theorem 8.1. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (5.1). Then*

$$\max_{i=1,\dots,n} \| U_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \leq C\left(N^{-1} \ln N\right)^2,$$

where the constant C is independent of the parameters ε_i.

Proof: Since

$$\| U_{i,k} - u_{i,k} \|_{\varepsilon_i,\Omega^N} \leq \| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} + \| u^*_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N},$$

the result follows by combining Lemmas (6.1) and (8.1).

Theorem 8.2. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (5.1). Then the following parameter uniform error estimate holds*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} \| U_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \leq C\left(N^{-1} \ln N\right)^2$$

where the constant C is independent of the parameters ε_i.

Proof: Since $\sigma_r \leq \frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}}$, $r = m, \dots, 1$, consider k satisfies, $1 \leq k \leq \frac{N}{4s}$ and $1 - \left(\frac{N}{4s}\right) \leq k \leq N$, $s = 1, \dots, n-1$ on a neighbourhood of the boundary layers.

Using the Cauchy Schwarz inequality and Theorem 8.1,

$$\begin{aligned}
\left|(U_{i,k} - u_{i,k})(x_k)\right| &= \left|\int_{\Omega_k} (U_{i,k} - u_{i,k})(s)ds\right| \\
&\leq \left(\frac{1}{\varepsilon_r}\int_{\Omega_k} 1^2 ds\right)^{\frac{1}{2}} \left(\varepsilon_r \int_{\Omega_k} \left|(U_{i,k} - u_{i,k})'(s)\right| 2ds\right)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{\sigma_r}{\varepsilon_r}} \| U_{i,k} - u_{i,k} \|_{\varepsilon_r,\Omega^N} \\
&\leq CN^{-1}(\ln N)^{\frac{5}{2}}. \qquad (8.1)
\end{aligned}$$

On the other hand, Suppose that k satisfies $\frac{N}{4} \leq k \leq \frac{3N}{4}$, outside the boundary layers, $h_k \geq \frac{1}{N}$ and so

$$\begin{aligned}
\left|(U_{i,k} - u_{i,k})(x_k)\right|^2 &\leq N h_k \left|(U_{i,k} - u_{i,k})(x_k)\right|^2 \\
&\leq N \sum_{k=\frac{N}{4}}^{\frac{3N}{4}} h_k \left|(U_{i,k} - u_{i,k})(x_k)\right|^2 \\
&\leq N \| U_{i,k} - u_{i,k} \|^2_{l^2(\Omega^N)}.
\end{aligned}$$

Using Theorem (8.2) then leads to

$$\begin{aligned}
\| \left(U_{i,k} - u_{i,k}\right)(x_k) \| &\leq N(N^{-1}\ln N)^2 \| U_{i,k} - u_{i,k} \|_{l^2(\Omega^N)} \\
&\leq CN^{-\frac{1}{2}} (\ln N)^2. \qquad (8.2)
\end{aligned}$$

Combining (8.1) and (8.2) completes the proof.

6. Numerical Illustrations

Example 9.1. Consider the BVP

$$-E\vec{u}''(x) + A(x)\vec{u} = \vec{f}(x), \quad for\ x \in (0,1), \qquad \vec{u}(0) = \vec{0}, \vec{u}(1) = \vec{0}$$

Where $E = diag\,(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $A = \begin{pmatrix} 6 & -1 & 0 \\ -1 & 5(1+x) & -1 \\ -1 & -(1+x^2) & 6+x \end{pmatrix}$,

$\vec{f} = \left(e^x, 2, 1 + x^2\right)^T$. For various values of

$\varepsilon_1, \varepsilon_2, \varepsilon_3 \;\; N = 8k, \quad k = 2^r, \quad r = 3, \cdots, 8, \quad and \;\; \alpha = 2.0,$

Using the general methods from [6], the $\vec{\varepsilon}$-uniform order of convergence and the $\vec{\varepsilon}$-uniform error constant are computed by applying fitted mesh method to the example 9.1 shown in the figure1. In the following table outlines the conclusions.

$Values\ of \;\; D_{\varepsilon}^N, \quad D^N, \quad p^N, \quad p^* \; and \; C_{p^*}^N \; for\ \varepsilon_1 = \dfrac{\eta}{32}, \quad \varepsilon_2 = \dfrac{\eta}{16}, \quad \varepsilon_3 = 1.0.$

η	Number of mesh points N				
	64	128	256	512	1024
2^0	0.7544E-03	0.1717E-03	0.6677E-04	0.2797E-04	0.1303E-04
2^{-2}	0.1786E-02	0.2975E-03	0.1115E-03	0.4510E-04	0.2050E-04
2^{-4}	0.3974E-02	0.7429E-03	0.1842E-03	0.7169E-04	0.3064E-04
2^{-6}	0.8120E-02	0.1769E-02	0.3029E-03	0.1139E-03	0.4607E-04
2^{-8}	0.1492E-01	0.3948E-02	0.7378E-03	0.1837E-03	0.7132E-04
2^{-10}	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
2^{-12}	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
2^{-14}	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
D^N	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
P^N	0.1329E+01	0.1389E+01	0.1453E+01	0.1473E+01	
C_P^N	0.9233E+00	0.9053E+00	0.7898E+00	0.5031E+00	0.5032E+00
Computed order of $\vec{\varepsilon}$ uniform convergence, p^* = 1.329					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N$ = 0:9233					

3. REFERENCES

[1] H. -G. Roos, M. Stynes, L.Tobiska, Numerical methods for singularly perturbed differential equations, Springer Verlag, 1996.

[2] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems (Revised Edition),World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong, 2012.

[3] P. A. Farrell, A. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, Robust Computational Techniques for Boundary Layers, Applied Mathematics & Mathematical Computation (Eds. R. J. Knops & K. W. Morton), Chapman & Hall/CRC Press, 2000.

[4] P. A. Farrell, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, Singularly perturbed differential equations with discontinuous source terms, Proceedings of Workshop '98, Lozenetz, Bulgeria, Aug 27-31, 1998.

[5] P. A. Farrell, E. O'Riordan, G. I. Shishkin, A class of singularly perturbed semi-linear differential equations with interior layers, Mathematics of Computation, vol. 74, No. 252, June 7, 1759-1776, 1998.

[6] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, S.Wang, A parameter-uniform Schwarz method for a Singularly Perturbed reaction-diffusion problem with an interior layer, Applied Numerical Mathematics, 35, 323-337, 2000.

[7] T. Linss, N.Madden, Analysis of a FEM for a coupled system of singularly perturbed reaction diffusion equations, Numer Algor 50:283291, 2009.

[8] T. Linss, N. Madden, A finite element analysis of a coupled system of singularly perturbed reaction diffusion equations.,Appl. Math. Comput. 148(3), 869880, 2004.

[9] M.Paramasivam, S.Valarmathi, J. J. H. Miller, Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system, Mathematical Communications, vol.15, No.2, 2010.
[10] M.Paramasivam, S.Valarmathi, J. J. H. Miller, Second order parameter-uniform convergence for a finite difference method for a partially singularly perturbed linear reaction-diffusion system, Mathematical Communications, vol.18, No.2, 2013.
[11] M.Vinoth, M. Paramasivam, Parameter-uniform convergence of a finite element method for a partially singularly perturbed linear reaction-diffusion system, filomat, communicated.