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Solutions Of Negative Pell's Equation Involving Interprimes

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Abstract: Many researchers have been devoted to finding the solutions (η, ζ) in the set of non-negative integers, of Diophantine equation (Pell Equation) of the type $\eta^2 = D\zeta^2 \pm \alpha$, where the value α is fixed positive integers. In this article, we look for non-trivial integer solutions to two negative Pell equations $x^2 = 26y^2 - 9^t$, and $x^2 = 50y^2 - 34^t$, $t \in N$ for the different choices of t particular by $(i)\ t = 1$, $(ii)\ t = 3$, $(iii)\ t = 5$, $(iv)\ t = 2k$, $(v)\ t = 2k + 5$, $\forall\ k \in N$. Additionally, recurrence relations on the solutions are obtained.

KeyWords: Diophantine Equation, Integral solution, Pell's Equation, Brahma Gupta Lemma, Interprime.
2010 Mathematics Subject Classifications: 11D09, 11D61, 11D72.

1. INTRODUCTION

Number theory is the branch of Mathematics concerned with studying the properties and relations of integers. Many of these problems are concerned with the properties of prime numbers. Number theory includes the different aspects of natural numbers and their extensions in various fields of Mathematics and Science. There are number of branches of number theory including algebraic number theory, analytic number theory, geometric number theory, combinatorial number theory, computational number theory, probabilistic number theory and so on. Number theory also includes the study of irrational numbers, transcendental numbers, continued fractions and Diophantine equation [4, 5, 6].

Diophantine equations are named after the Greek Mathematician *Diophantus* of Alexandria, whose book *Arithmetica* included a study of such equations [13 - 16]. A Diophantine equation is a numerical polynomial equation that has more than one unknown quantity and integer coefficients, and for which a solution is sought in integers [1 - 5].For example $23x + 21y + 7 = 0$ is a Diophantine equation, where x and y are the unknown quantities. Famous examples of Diophantine equations include Pythagoras' theorem and Fermat's last theorem. Individual Diophantine problems were studied by such great Mathematicians like Euler, Gauss and Fermat.

Pierre de Fermat was a 17th century French lawyer and amateur Mathematician. He is often credited with founding modern number theory and he made some of the greatest advances in the history of Mathematics. Fermat considered Pythagoras' theorem, which states that, for every right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides which is rewritten as $x^2 + y^2 = z^2$.Fermat asked himself what would happen if, instead of being squared, the numbers were cubed or raised to the power of

4, 5, or any other higher number. This led to Fermat's last theorem which states that it is impossible to find a non-zero integer solution to the equation $x^n + y^n = z^n$ if n is any integer greater than 2.The Indian contribution is also there such as Aryabhata (499) who gave the first explicit description of the general integer solutions of the linear Diophantine equation which occurs in his text $Aryabhata$.Brahmagupta in 628 handled more difficult Diophantine equations including forms of $Pellian$ Equation such as $y^2 = 61x^2 + 1$. Bhaskara II in 1150, was using a modified version of $Brahmagupta's\ Charavate$ method to find the general solution to quadratic Diophantine equations.

Diophantine equations are numerically rich because of their variety. There is no universal method available to know whether a Diophantine equation has a solution or for finding all solutions, if it exists. There are but very few Diophantine problems for each of which the complete solution is known. For example, it is possible to derive all the triplets of integers (*x*, *y*, *z*) that satisfy the equation $x^2 + y^2 = z^2$. There are several Diophantine equations that have no solutions, trivial solutions, finitely many solutions or an infinite number of solutions. For example, $xy = x + 2y + 2$ has finite number of solutions and they are$(0,-1), (-2,0), (1,-3), (3,5), (4,3), (6,2)$. The binary quadratic equation $2(x + y) + xy = x^2 - y^2$ representing a hyperbola has infinitely many solutions. The Pellian equation $y^2 = 3x^2 - 1$ has no solution in integers.

The Pell's equation is the equation $x^2 = dy^2 + 1$ to be solved in positive integer x, y for a non – zero integers d. For example, for $d = 5$ one can take $x = 9, y = 4$. We shall always assume that d is positive but not a square, since otherwise there are clearly no solutions. Pell's equation has an extra ordinarily rich history to which Weil [7] is the best guide. A particularly lucid exposition of method of solving the Pell equation is found in Euler's algebra [15].

In mathematics, an $interprime$ is the average of two consecutive odd primes. For example, 9 is an $interprime$ because it is the average of 7 and 11. The first $interprimes$ are: $4, 6, 9, 12, 15, \cdots$. Here using $Interprime$ $26, 9, 50$ & 34 we form a Pell's equations $x^2 = 26y^2 - 9^t$, $x^2 = 50y^2 - 34^t$, $t \in \boldsymbol{N}$ and search for its non- trivial integer solutions.

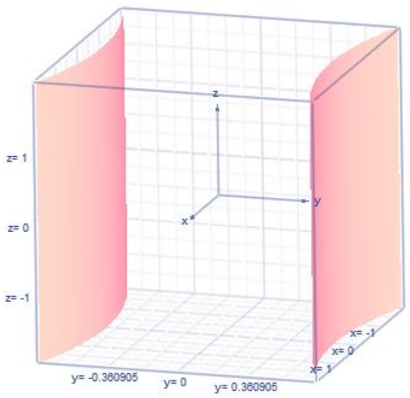
This communication concerns with two negative Pell's equations $x^2 = 26y^2 - 9^t$, and $x^2 = 50y^2 - 34^t$, where $t \in \boldsymbol{N}$ and infinitely many positive integer solutions are obtained for the choices of t given by $(i)\ t = 1$, $(ii)\ t = 3$ $(iii) t = 5$ $(iv)\ t = 2k$ and (v) $t = 2k + 5$. A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are derived.

2. METHOD OF ANALYSIS

2.1. Diophantine Equation $x^2 = 26y^2 - 9^t$:

In this section concerns with the negative Pell's equation $x^2 = 26y^2 - 9^t, t \in \boldsymbol{N}$, and infinitely many positive integer solutions are obtained for the choices of t given by $(i)\ t = 1$, $(ii)\ t = 3$ $(iii) t = 5$ $(iv)\ t = 2k,\ k \in N$ and $(v) t = 2k + 5,\ k \in N$.

Further recurrence relations on the solutions are derived. Pictorial representation of the equation $x^2 = 26y^2 - 9^t,\ t \in N$:



2.1.1: Choice 1: $t = 1$

The Pell equation is

$$x^2 = 26y^2 - 9 \qquad (1)$$

Let (x_0, y_0) be the initial solution of (1) given by

$$x_0 = 15\,;\ \ y_0 = 3$$

To find the other solutions of (1), consider the Pell equation

$$x^2 = 26y^2 + 1$$

whose initial solution $(\widetilde{x_n},\ \widetilde{y_n})$ is given by

$$\widetilde{x_n} = \frac{1}{2} f_n$$

$$\widetilde{y_n} = \frac{1}{2\sqrt{26}}\ g_n$$

where $f_n = \left(51 + 10\sqrt{26}\right)^{n+1} + \left(51 - 10\sqrt{26}\right)^{n+1}$

$g_n = \left(51 + 10\sqrt{26}\right)^{n+1} - \left(51 - 10\sqrt{26}\right)^{n+1},\ n = 0,1,2,\cdots$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (1) are obtained as

$$x_{n+1} = \tfrac{1}{2}\left[15 f_n + 3\sqrt{26}\, g_n\right] \qquad (2)$$

$$y_{n+1} = \tfrac{1}{2\sqrt{26}}\ \left[3\sqrt{26}\, f_n + 15 g_n\right] \qquad (3)$$

The recurrence relation satisfied by the solutions of (1) are given by

$$x_{n+2} - 102x_{n+1} + x_n = 0$$

$$y_{n+2} - 102y_{n+1} + y_n = 0$$

2.1.2 Choice 2: $t = 3$

The Pell equation is

$$x^2 = 26y^2 - 729 \qquad (4)$$

Let (x_0, y_0) be the initial solution of (4) given by

$$x_0 = 135;\ \ y_0 = 27$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (4) are obtained as

$x_{n+1} = \tfrac{1}{2}\left[135 f_n + 27\sqrt{26}\, g_n\right]$ (5) $y_{n+1} = \tfrac{1}{2\sqrt{26}}\left[27\sqrt{26}\, f_n + 135\, g_n\right]$ (6)

The recurrence relations satisfied by the solutions of (4) are given by

$$x_{n+2} - 102x_{n+1} + x_n = 0$$

$$y_{n+2} - 102y_{n+1} + y_n = 0$$

2.1.3 Choice 3: $t = 5$

The Pell equation is

$$x^2 = 26y^2 - 59049 \quad (7)$$

Let (x_0, y_0) be the initial solution of (7) given by

$$x_0 = 1215; \quad y_0 = 243$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (7) are obtained as

$$x_{n+1} = \frac{1}{2}\left[1215\, f_n + 243\sqrt{26}\, g_n\right] \quad (8) \qquad y_{n+1} = \frac{1}{2\sqrt{26}}\left[243\sqrt{26}\, f_n + 1215\, g_n\right] \quad (9)$$

The recurrence relations satisfied by the solutions of (7) are given by

$$x_{n+2} - 102x_{n+1} + x_n = 0$$
$$y_{n+2} - 102y_{n+1} + y_n = 0$$

2.1.4 Choice 4: $t = 2k,\ k > 0$

The Pell equation is

$$x^2 = 26y^2 - 9^{2k},\ k > 0 \quad (10)$$

Let (x_0, y_0) be the initial solution of (10) given by

$$x_0 = 9^k.5\ ; \quad y_0 = 9^k.1$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (10) are obtained as

$$x_{n+1} = \frac{9^k}{2}\left[5f_n + \sqrt{26}\, g_n\right] \quad (11) \qquad y_{n+1} = \frac{9^k}{2\sqrt{26}}\left[\sqrt{26}\, f_n + 5\, g_n\right] \quad (12)$$

The recurrence relations satisfied by the solutions of (10) are given by

$$x_{n+2} - 102x_{n+1} + x_n = 0$$
$$y_{n+2} - 102y_{n+1} + y_n = 0$$

2.1.5 Choice 5: $t = 2k + 5,\ k > 0$

The Pell equation is

$$x^2 = 26y^2 - 9^{2k+5},\ k > 0 \quad (13)$$

Let (x_0, y_0) be the initial solution of (13) given by

$$x_0 = 9^{k-1}.10935\ ; \quad y_0 = 9^{k-1}.2187$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (13) are obtained as

$$x_{n+1} = \frac{9^{k-1}}{2}\left[10935\, f_n + 2187\sqrt{26}\, g_n\right] \quad (14)$$

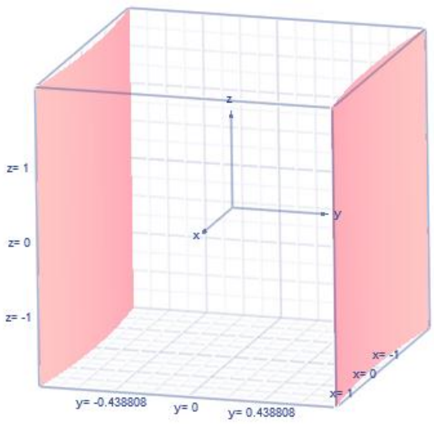
$$y_{n+1} = \frac{9^{k-1}}{2\sqrt{26}}\left[2187\sqrt{26}\, f_n + 10935\, g_n\right] \quad (15)$$

The recurrence relations satisfied by the solutions of (13) are given by

$$x_{n+2} - 102x_{n+1} + x_n = 0$$
$$y_{n+2} - 102y_{n+1} + y_n = 0$$

2.2. Diophantine Equation $x^2 = 50y^2 - 34^t$

In this section concerns with the negative Pell's equation $x^2 = 50y^2 - 34^t$, where $t \in \boldsymbol{N}$, and infinitely many positive integer solutions are obtained for the choices of t given by $(i)\ t = 1,\ (ii)\ t = 3\ (iii) t = 5\ (iv)\ t = 2k$ and $(v) t = 2k + 5$. Further recurrence relations on the solutions are derived. Pictorial representation of the equation $x^2 = 50y^2 - 34^t,\ t \in N$:



2.2.1: Choice 1: $t = 1$

The Pell equation is

$$x^2 = 50y^2 - 34 \qquad (16)$$

Let (x_0, y_0) be the initial solution of (16) given by

$$x_0 = 4\,;\;\; y_0 = 1$$

To find the other solutions of (16), consider the Pell equation

$$x^2 = 50y^2 + 1$$

whose initial solution $(\widetilde{x_n},\ \widetilde{y_n})$ is given by

$$\widetilde{x_n} = \frac{1}{2}\, f_n$$

$$\widetilde{y_n} = \frac{1}{2\sqrt{50}}\; g_n$$

where

$$f_n = \left(99 + 14\sqrt{50}\right)^{n+1} + \left(99 - 14\sqrt{50}\right)^{n+1}$$

$$g_n = \left(99 + 14\sqrt{50}\right)^{n+1} - \left(99 - 14\sqrt{50}\right)^{n+1}, \quad n = 0,1,2,\cdots$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (16) are obtained as

$$x_{n+1} = \tfrac{1}{2}\left[4f_n + \sqrt{50}\, g_n\right] \qquad (17) \qquad y_{n+1} = \tfrac{1}{2\sqrt{50}}\left[\sqrt{50}\, f_n + 4g_n\right]$$

(18)

The recurrence relation satisfied by the solutions of (16) are given by

$$x_{n+2} - 198x_{n+1} + x_n = 0$$

$$y_{n+2} - 198y_{n+1} + y_n = 0$$

2.2.2 Choice 2: $t = 3$

The Pell equation is

$$x^2 = 50y^2 - 34^3 \qquad (19)$$

Let (x_0, y_0) be the initial solution of (19) given by

$$x_0 = 664; \quad y_0 = 98$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (19) are obtained as

$$x_{n+1} = \tfrac{1}{2}\left[664f_n + 98\sqrt{50}\, g_n\right] \qquad (20)$$

$y_{n+1} = \frac{1}{2\sqrt{50}}\left[98\sqrt{50}\, f_n + 664\, g_n\right]$ (21)

The recurrence relations satisfied by the solutions of (19) are given by

$$x_{n+2} - 198x_{n+1} + x_n = 0$$
$$y_{n+2} - 198y_{n+1} + y_n = 0$$

2.2.3 Choice 3: $t = 5$

The Pell equation is

$x^2 = 50y^2 - 34^5$ (22)

Let (x_0, y_0) be the initial solution of (23) given by

$$x_0 = 22576; \quad y_0 = 3332$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (22) are obtained as

$x_{n+1} = \frac{1}{2}\left[22576\, f_n + 3332\sqrt{50}\, g_n\right]$ (23) $y_{n+1} = \frac{1}{2\sqrt{50}}\left[3332\sqrt{50}\, f_n + 22576\, g_n\right]$
(24)

The recurrence relations satisfied by the solutions of (22) are given by

$$x_{n+2} - 198x_{n+1} + x_n = 0$$
$$y_{n+2} - 198y_{n+1} + y_n = 0$$

2.2.4 Choice 4: $t = 2k,\ k > 0$

The Pell equation is

$x^2 = 50y^2 - 34^{2k},\ k > 0$ (25)

Let (x_0, y_0) be the initial solution of (25) given by

$$x_0 = 34^k.7\,; \quad y_0 = 34^k.1$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (25) are obtained as

$x_{n+1} = \frac{34^k}{2}\left[7f_n + \sqrt{50}\, g_n\right]$ (26)

$y_{n+1} = \frac{34^k}{2\sqrt{50}}\left[\sqrt{50}\, f_n + 7\, g_n\right]$ (27)

The recurrence relations satisfied by the solutions of (25) are given by

$$x_{n+2} - 198x_{n+1} + x_n = 0$$
$$y_{n+2} - 198y_{n+1} + y_n = 0$$

2.2.5 Choice 5: $t = 2k + 5,\ k > 0$

The Pell equation is

$x^2 = 50y^2 - 34^{2k+5},\ k > 0$ (28)

Let (x_0, y_0) be the initial solution of (28) given by

$x_0 = 97^{k-1}.203584; y_0 = 34^{k-1}.43352$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (28) are obtained as

$x_{n+1} = \frac{34^{k-1}}{2}\left[203584\, f_n + 43352\sqrt{50}\, g_n\right]$ (29) $y_{n+1} = \frac{34^{k-1}}{2\sqrt{50}}\left[43352\sqrt{50}\, f_n + 203584\, g_n\right]$ (30)

The recurrence relations satisfied by the solutions of (28) are given by

$$x_{n+2} - 198x_{n+1} + x_n = 0$$
$$y_{n+2} - 198y_{n+1} + y_n = 0$$

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3. CONCLUSION

Solving a Pell's equation using the above method provides powerful tool for finding solutions of equations of similar type. Neglecting any time consideration it is possible using current methods to determine the solvability of Pell-like equation.

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