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Existence results for the Solutions of Fractional Evolution Equations with Non-instantaneous Impulses

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***Abstract:* In this article, we are discussing a set of sufficient conditions for existence mild solutions of fractional semi-linear evolution equation with non-instantaneous impulses with classical and non-local conditions using the concept of generators and generalized fixed point theorem. Illustrations provided to validate our results.**

***Keywords*: Existence of solution, Fractional evolution equation, Fixed point theorem, Non-instantaneous impulses.**

1. INTRODUCTION:

Recently, Shah et. al. [1] discussed the existence and uniqueness of classical solution for the impulsive evolution equation defining the concept of generators.This paper derive set of sufficient conditions for the existence of mild solution of the semi-linear evolution equation.

$$\begin{aligned} {}^cD^\alpha u(t) &= Au(t) + f(t, u(t)), \quad t \in [s_k, t_{k+1}),\ k = 1, 2, \cdots, p \\ u(t) &= g_k(k, u(t)),\ t \in [t_k, s_k) \end{aligned} \tag{1.1}$$

over the interval $[0, T]$ with local condition $u(0) = u_0$ and non-local condition $u(0) = u_0 + h(u)$ over the interval $[0, T]$ in a Banach space $\mathcal{U}$. Here $A : \mathcal{U} \to \mathcal{U}$ is linear operator, $f : [0, T] \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is nonlinear function and $g_k : [0, T] \times \mathcal{U}$ are set of nonlinear functions applied in the interval $[t_k, s_k)$ for all $i = 1, 2, \cdots, p$.

There are various problems in physics, engineering and biological sciences, having changes for a particular moment of time or small amount of time are well explained in terms of impulsive differential equations. Therefore the problems like removal of insertion of biomass, populations of species with abrupt changes, abrupt harvesting, and various problems containing abrupt changes are modelled into instantaneous impulsive differential equations [2–10]. The qualitative properties like existence, uniqueness and asymptotic behavior of impulsive differential equations using various techniques have been studied by many researchers [11–17]. Sometimes, changes may be for small duration of time. This type of problems are model into non-instantaneous impulsive equations. Details of the non-instantaneous impulsive equations are found in [18].

On the other hand, the fractional calculus and fractional evolution equations are became one of the popular branch of the applied mathematics as the fractional derivative operators have infinite degree of freedom and due to this many of the problems in physics, biology and economics are well approximated using fractional differential equations rather than integer order differential equations. Applications of fractional order models are found in the [19–30]. Existence of the mild solution of the fractional equations without and with instantaneous impulses are found in the articles [35–39] and reference their in. Study of solutions of fractional equations with non-instantaneous impulse are found in the articles [40–42] and reference their in.

2. PRILIMINARIES

Basic definitions and theorems of fractional calculus and functional analysis are discuss in this section, which will help us to prove our main results.

Definition 2.1. *[33] The Riemann-Liouville fractional integral operator of $\beta > 0$, of function $h \in L_1(\mathbb{R}_+)$ is defined as*

$$J^{\beta}_{t_0+}h(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-q)^{\beta-1}h(q)dq,$$

provided the integral on right side exist. Where, $\Gamma(\cdot)$ is gamma function.

Definition 2.2. *[34] The Caputo fractional derivative of order $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, is defined as*

$${}^{c}D^{\beta}_{t_0+}h(t) = \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-q)^{n-\beta-1}\frac{d^n h(q)}{dq^n}dq$$

where, the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3. *[1] The families of operators $T(t), T_{\alpha}(t) : \mathcal{U} \to \mathcal{U}$, $t \geq 0$ are generated by a linear operator $A : \mathcal{U} \to \mathcal{U}$ satisfies the following properties:*

(1) $T(0) = I$ where, I is identity operator

(2) $T(t)$ satisfies the linear fractional equation ${}^{c}D^{\alpha}u(t) = A(t)u(t)$ in Banach space $\mathcal{U}$

(3) $\lim_{\alpha\to 1} T_{\alpha}(t) = T(t)$

Theorem 2.1. *(Banach Fixed Point Theorem) [40] Let E be closed subset of a Banach Space $(\mathcal{X}, \|\cdot\|)$ and let $T : E \to E$ contraction then, T has unique fixed point in E.*

Theorem 2.2. *(Krasnoselskii's Fixed Point Theorem) [40] Let E be closed convex nonempty subset of a Banach Space $(\mathcal{X}, \|\cdot\|)$ and P and Q are two operators on E satisfying:*

(1) $Pu + Qv \in E$, whenever $u, v \in E$,

(2) P is contraction,

(3) Q is completely continuous

then, the equation $Pu + Qu = u$ has unique solution.

3. EQUATION WITH CLASSICAL CONDITIONS

This section, derived the existence and uniqueness results for the fractional evolution equation:

$$\begin{aligned} {}^cD^\alpha u(t) &= Au(t) + f(t,u(t)), && t \in [s_k, t_{k+1}), \quad k = 1,2,\cdots,p \\ u(t) &= g_k(t,u(t)), && t \in [t_k, s_k) \quad k = 1,2,\cdots,p \\ u(0) &= u_0 \end{aligned} \tag{3.1}$$

over the interval $[0,T]$ in the Banach space $\mathcal{U}$,.

Definition 3.1. *The function $u(t)$ is called mild solution of the impulsive fractional equation* (3.1) *over the interval* $[0,T]$, *if $u(t)$ satisfies the integral equation* (3.1).

$$u(t) = \begin{cases} T(t)u_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(t,u(s))ds, & t \in [0,t_1) \\ g_k(t,u(t)), & t \in [t_k, s_k) \\ T(t-s_k)g_k(s_k,u(s_k)) + \int_{s_k}^t (t-s)^{\alpha-1} T_\alpha(t-s) f(t,u(s))ds, & t \in [s_k, t_{k+1}) \end{cases} \tag{3.2}$$

where, the operators $T(t)$ and $T_\alpha(t)$ are the generators of the linear part A.

Theorem 3.1. *(Existence and uniqueness Theorem) If assumptions*

(A1) The families of operators $T(t)$ and $T_\alpha(t)$ generated by the operator $A(t)$ are continuous and bounded over $[0,T]$. That is, there exist positive constants M and M_α such that $||T(t)|| \leq M$ and $||T_\alpha(t)|| \leq M_\alpha$ for all $t \in [0,T]$.

(A2) The nonlinear function f is continuous with respect to t and there exist r_0 such that F Lipchitz continuous with respect to u in $B_{r_0} = \{u \in \mathcal{U}; ||u|| \leq r_0\}$. That is, there exist positive constant L such that $||f(t,u) - f(t,v)|| \leq L||u - v||$ for all $t \in [0,T]$ and $u \in B_{r_0}$.

(A3) The functions $g_k : [t_k, s_k] \times \mathcal{U}$ are continuous and there exist a positive constants $0 < G_k < 1$ such that $||g_k(t,u(t)) - g_k(t,v(t))|| \leq G_k||u - v||$.

are satisfied, then the semi-linear fractional evolution equation with not-instantaneous impulses (3.1) *has unique mild solution.*

Proof. Define the operator $\mathcal{F}$ on $\mathcal{U}$ by

$$\mathcal{F}u(t) = \begin{cases} \mathcal{F}_1 u(t), & t \in [0,t_1) \\ \mathcal{F}_{2k} u(t), & t \in [t_k, s_k) \\ \mathcal{F}_{3k} u(t), & t \in [s_k, t_{k+1}) \end{cases}$$

where, $\mathcal{F}_1$, $\mathcal{F}_{2k}$ and $\mathcal{F}_{3k}$ are

$$\begin{aligned} \mathcal{F}_1 u(t) &= T(t)u_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(t,u(s))ds, t \in [0,t_1) \\ \mathcal{F}_{2k} u(t) &= g_k(t,u(t)), t \in [t_k, s_k) \\ \mathcal{F}_{3k} u(t) &= T(t-s_k)g_k(s_k,u(s_k)) \int_{s_k}^t (t-s)^{\alpha-1} T_\alpha(t-s) f(t,u(s))ds, t \in [s_k, t_{k+1}) \end{aligned}$$

for all $k = 1,2,\cdots p$.

In view of this operator $\mathcal{F}$, the equation (3.2) has unique solution if and only if the operator equation $u(t) = \mathcal{F}u(t)$ has unique solution. This is possible if and only if each of $u(t) = \mathcal{F}_1 u(t)$, $u(t) = \mathcal{F}_{2k}u(t)$ and $u(t) = \mathcal{F}_{3k}u(t)$ has unique solution over the interval $[0, t_1)$, $[t_k, s_k)$ and $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$ respectively.
For all $t \in [0, t_1)$ and $u, v \in B_{r_0}$,

$$||\mathcal{F}_1^{(n)}u(t) - \mathcal{F}_1^{(n)}v(t)|| \leq \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} (t-\tau_1)^{\alpha-1}(\tau_1-\tau_2)^{\alpha-1}\cdots(\tau_{n-1}-s)^{\alpha-1}||T_\alpha(t-\tau_1)|| \\ ||T_\alpha(\tau_1-\tau_2)||\cdots||T_\alpha(\tau_{n-1}-s)||||f(s,u(s)) - f(s,v(s))||ds d\tau_{n-1}\cdots d\tau_1$$

By applying assumption (A1) and (A2) we get,

$$\begin{aligned} ||\mathcal{F}_1^{(n)}u(t) - \mathcal{F}_1^{(n)}v(t)|| &\leq \int_0^{t_1}\int_0^{t_1}\cdots\int_0^{t_1} t_1^{n(\alpha-1)} M_\alpha^n L||u-v||ds d\tau_{n-1}\cdots d\tau_1 \\ &\int_0^{t_1}(t_1-s)^{n-1}ds||u-v|| \\ &\leq \frac{t_1^{n\alpha}M_\alpha^n L}{n!}||u-v|| \\ &\leq c^*||u-v||. \end{aligned}$$

Considering maximum over interval $[0, t_1)$ we get $||\mathcal{F}_1^{(n)}u - \mathcal{F}_1^{(n)}v| \leq c^*||u-v|| \to 0$ for fixed t_1. Therefore there exist m such that $\mathcal{F}_1^{(m)}$ is contraction on B_{r_0}. Thus by general Banach contraction theorem the operator equation $u(t) = \mathcal{F}_1 u(t)$ has unique solution over the interval $[0, t_1)$.
For all $k = 1, 2, \cdots, p$, $t \in [t_k, s_k)$ and $u, v \in \mathcal{U}$ and assuming (A3)

$$||\mathcal{F}_{2k}u(t) - \mathcal{F}_{2k}v(t)|| = ||g_k(t,u(t)) - g_k(t,v(t))|| \leq G_k||u-v||.$$

Then $\mathcal{F}_{2k}$ is contraction and by Banach fixed point theorem the operator equation $u(t) = \mathcal{F}_{2k}u(t)$ has unique solution for the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$. This means for all $k = 1, 2, \cdots, p$, $u(t) = g_k(t, u(t))$ has unique solution for all $t \in [t_k, s_k)$. Lipschitz continuity of g_k leads to uniqueness of the solution at point s_k also.
For all $k = 1, 2, \cdots, p$, $t \in [s_k, t_{k+1})$ and $u, v \in B_{r-0}$,

$$||\mathcal{F}_{3k}^{(n)}u(t) - \mathcal{F}_{3k}^{(n)}v(t)|| \leq \int_{s_k}^t \int_{s_k}^{\tau_1} \cdots \int_{s_k}^{\tau_{n-1}} (t-\tau_1)^{\alpha-1}(\tau_1-\tau_2)^{\alpha-1}\cdots(\tau_{n-1}-s)^{\alpha-1}||T_\alpha(t-\tau_1)|| \\ ||T_\alpha(\tau_1-\tau_2)||\cdots||T_\alpha(\tau_{n-1}-s)||||f(s,u(s)) - f(s,v(s))||ds d\tau_{n-1}\cdots d\tau_1$$

Appalling assumption (A1) and (A2) and we get,

$$\begin{aligned} ||\mathcal{F}_{3k}^{(n)}u(t) - \mathcal{F}_{3k}^{(n)}v(t)|| &\leq \int_{s_k}^{t_{k+1}}\int_{s_k}^{t_{k+1}}\cdots\int_{s_k}^{t_{k+1}} (t_{k+1}-s_k)^{n(\alpha-1)} M_\alpha^n L||u-v||ds d\tau_{n-1}\cdots d\tau_1 \\ &\leq \frac{(t_{k+1}-s_k)^{n(\alpha-1)}M_\alpha^n L}{(n-1)!} \\ &\int_{s_k}^{t_{k+1}}(t_{k+1}-s)^{n-1}ds||u-v|| \leq \frac{(t_{k+1}-s_k)^{n\alpha}M_\alpha^n L}{n!}||u-v|| \\ &\leq c^*||u-v||. \end{aligned}$$

Considering supremum over interval $[s_k, t_{k+1})$ we get $||\mathcal{F}_{3k}^{(n)}u - \mathcal{F}_{3k}^{(n)}v| \leq c^*||u-v|| \to 0$ for fixed sub-interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$. Therefore there exist m such that $\mathcal{F}_{3k}^{(m)}$ is contraction on B_{r_0}. Thus by general Banach contraction theorem the operator equation $u(t) = \mathcal{F}_{3k}u(t)$ has unique solution over the interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$.
Hence, the operator equation $u(t) = \mathcal{F}u(t)$ has unique solution over the interval $[0, T]$ which is nothing but mild solution of the equation (3.1). This completes the proof of the theorem. □

Example 3.1. *The fractional order evolution equation:*

$$\begin{aligned} {}^cD^\alpha u(t,x) &= \frac{\partial^2 w}{\partial x^2}(t,x) + u\frac{\partial u}{\partial x}(t,x) + f(t,u(t,x)), && t \in [0,\frac{1}{3}) \cup [\frac{2}{3},1] \\ u(t,x) &= \frac{u(t,x)}{2(1+u(t,x))}, && t \in [\frac{1}{3},\frac{2}{3}) \end{aligned} \tag{3.3}$$

over the interval $[0,1]$ *with initial condition* $u(0,x) = u_0(x)$ *and boundary condition* $u(t,0) = u(t,2\pi) = 0$. *The domain of the operator* $Au = -\frac{\partial^2 u}{\partial x^2}$ *is* $\mathcal{D}(A) = \{z \in L^2[0,2\pi]/z$ *is twice differentiable and* $z(0) = z(2\pi) = 0\}$. *The function* $u(t)$ *is mild solution of* (3.3) *in the interval* $[0,1]$ *if* $u(t)$ *satisfies:*

$$u(t) = \begin{cases} T(t)u_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\{\frac{1}{2}\frac{\partial u^2}{\partial x} + f(s,u)\}ds, & t \in [0,\frac{1}{3}) \\ \frac{u(t,x)}{2(1+u(t,x))}, & t \in [\frac{1}{3},\frac{2}{3}) \\ T(t-2/3)\frac{u(2/3,x)}{2(1+u(2/3,x))} + \int_{2/3}^t (t-s)^{\alpha-1}T_\alpha(t-s)\{\frac{1}{2}\frac{\partial u^2}{\partial x} + f(s,u))\}ds, & t \in [\frac{2}{3},1] \end{cases} \tag{3.4}$$

Here,

$$T(t)z = \sum_{n=1}^{\infty} E_\alpha(-n^2t^\alpha) < z, \phi_n > \phi_n \text{ and } T_\alpha(t)z = \sum_{n=1}^{\infty} E_{\alpha,\alpha}(-n^2t^\alpha) < z, \phi_n > \phi_n$$

are the generators of the linear operator A. $\phi_n(x)$ *are orthogonal Fourier basis functions in* $L^2[0,2\pi]$.

1 *The generators* $T(t)$ *and* $T_\alpha(t)$ *are continuously differentiable with respect to* t. *Therefore there exists positive constants* M *and* M_α *such that* $||T(t)|| \leq M$ *and* $||T_\alpha(t)|| \leq M_\alpha$ *respectively.*

2 *The first non linear term in* (3.3) $\frac{1}{2}\frac{\partial u^2}{\partial x}$ *is composition of two continuous operators* $Pu = \frac{1}{2}\frac{\partial u}{\partial x}$ *and* $Qu = u^2$ *which are continuous with respect to* t *and Lipchitz continuous with respect to* u *in finite closed ball* B_{r_0} *as the operator* P *is linear and the partial derivative of* Q *with respect to* u *exist for every* u. *Moreover* P *and* Q *are differentiable with respect to arguments* t *and* u.

3 *The impulse* $g(t,z(t)) = \frac{z(t)}{2(1+z(t))}$ *is continuous with respect to* t *and Lipchitz continuous with respect to* z *with Lipschitz constant* $G* = 1/2 < 1$.

Therefore, by theorem (3.1) the equation (3.3) *has unique solution over* $[0,1]$.

4. EQUATION WITH NON-LOCAL CONDITIONS

This section, derived the existence results for the fractional evolution equation:

$$\begin{aligned} {}^cD^\alpha u(t) &= Au(t) + f(t,u(t)), && t \in [s_k,t_{k+1}), \quad k = 1,2,\cdots,p \\ u(t) &= g_k(t,u(t)), && t \in [t_k,s_k) \quad k = 1,2,\cdots,p \\ u(0) &= u_0 + h(u) \end{aligned} \tag{4.1}$$

over the interval $[0,T]$ in the Banach space $\mathcal{U}$.

Definition 4.1. *The function* $u(t)$ *is called mild solution of the impulsive fractional equation* (4.1) *over the interval if* $u(t)$ *satisfies the integral equation* (4.1).

$$u(t) = \begin{cases} T(t)[u_0 + h(u)] + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(t,u(s))ds, & t \in [0,t_1) \\ g_k(t,u(t)), & t \in [t_k,s_k) \\ T(t-s_k)g_k(s_k,u(s_k)) + \int_{s_k}^t (t-s)^{\alpha-1}T_\alpha(t-s)f(t,u(s))ds, & t \in [s_k,t_{k+1}) \end{cases} \tag{4.2}$$

Theorem 4.1. *(Existence Theorem) If assumptions,*

(B1) The families of operators $T(t)$ and $T_\alpha(t)$ generated by the operator $A(t)$ are continuous and bounded over $[0,T]$. That is, there exist positive constants M and M_α such that $||T(t)|| \leq M$ and $||T_\alpha(t)|| \leq M_\alpha$ for all $t \in [0,T]$.

(B2) The function $f(t,\cdot)$ is continuous and $f(\cdot,u)$ is measurable on $[0,T]$. Also there exist $\beta \in (0,\alpha)$ with $m_f \in L^{\frac{1}{\beta}}([0,T],\mathbb{R})su$ such that $|f(t,u)| \leq m_f(t)||u||$ for all $u,v \in \mathcal{U}$.

(B3) The functions $g_k : [t_k, s_k] \times \mathcal{U}$ are continuous and there exist a positive constants $0 < G_k < 1$ such that $||g_k(t,u(t)) - g_k(t,v(t))|| \leq G_k||u - v||$ for all k.

(B4) The operator $h : \mathcal{U} \to \mathcal{U}$ is Lipschitz continuous with respect to u with Lipschitz constant $0 < H \leq 1$.

are satisfied, then the non-local semi-linear fractional order evolution equation (4.1) has mild solution over the interval $[0,T]$ provided $MH < 1$ and $MG < 1$.

Proof. Assuming (B1), for all $u \in B_k = \{u \in \mathcal{U} : ||u|| \leq k\}$ for any positive constant k. Therefore,

$$|T(t)(u_0 + h(u))| \leq M(|u_0| + H||u|| + |h(0)|), \tag{4.3}$$

and assuming (B2) one can easily shows that $(t-s)^{\alpha-1} \in L^{\frac{1}{1-\beta}}[0,t]$ for all $t \in [0,T]$ and $\beta \in (0,\alpha)$. Let$b = \frac{\alpha-1}{1-\beta} \in (-1,0)$ and set $M_1 = ||m_f||_{L^{\frac{1}{\beta}}}$. Using assumption (B2) and Holder's inequality for $t \in [0,T]$ we get,

$$\begin{aligned}\int_0^t |(t-s)^{\alpha-1}T_\alpha(t-s)f(s,u(s))|ds &\leq M_\alpha\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta} M_1||u|| \\ &\leq \frac{M_\alpha M_1}{(1+b)^{1-\beta}}T^{(1+b)(1-\beta)}||u||.\end{aligned} \tag{4.4}$$

For $t \in [0,t_1)$ and for positive r we define F_1 and F_2 on B_r as,

$$\begin{aligned}F_1u(t) &= T(t)(u_0 + h(u)) \\ F_2u(t) &= \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(t,u(s))ds\end{aligned}$$

then, $u(t)$ is mild solution of the semi-linear fractional evolution equation if and only if the operator equation $u = F_1u + F_2u$ has solution for $u \in B_r$ for some r. Therefore the existence of a mild solution of (4.1) over the interval $[0,t_1)$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}.

Step:1 $||F_1u + F_2v|| \leq r_0$ for some positive r_0.

Let $u, v \in B_{r_0}$, choose

$$r_0 = M\frac{|u_0| + |h(z)|}{1 - MH} + \frac{M_\alpha M_1}{(1-MH)(1+b)^{1-\beta}}t_1^{(1+b)(1-\beta)},$$

and consider

$$\begin{aligned}|F_1u(t) + F_2v(t)| &\leq \left|T(t)(u_0 + h(u))\right| + \left|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(t,v(s))ds\right| \\ &\leq M(|u_0| + H||u|| + |h(0)|) + \frac{M_\alpha M_1}{(1+b)^{1-\beta}}t_1^{(1+b)(1-\beta)}||v|| \\ &\quad \text{(using inequalities (4.3) and (4.4))} \\ &\leq r_0 \quad (\text{since, } MH < 1).\end{aligned}$$

Therefore, $||F_1u + F_2v|| \leq r_0$ for every pair $u, v \in B_{r_0}$.

Step: 2 F_1 is contraction on B_{r_0}.
For any $u, v \in B_{r_0}$ and $t \in [0, t_1)$, we have $|F_1u(t) - F_1v(t)| \leq MH||u - v||$. Taking supremum over $[0, t_1)$, $||F_1u - F_1v|| \leq MH||u - v||$. Since, $MH < 1$, F_1 is contraction.
Step: 3 F_2 is completely continuous operator on B_{r_0}.
Let the sequence $\{u_n\}$ in B_{r_0} converging to $u \in B_{r_0}$ then,

$$\begin{aligned}|F_2u_n(t) - F_2u(t)| &\leq \int_0^t (t-s)^{\alpha-1}|T_\alpha(t-s)||f(s,u_n(s)) - f(s,u(s))|ds\\ &\leq M_\alpha \int_0^t (t-s)^{\alpha-1} \sup_{s\in[0,t_1)} |f(s,u_n(s)) - f(s,u(s))|ds\\ &\leq \frac{M_\alpha t_1^\alpha}{\alpha} \sup_{s\in[0,t_1)} |f(s,u_n(s)) - f(s,u(s))|,\end{aligned}$$

which implies,

$$||F_2u_n - F_2u|| \leq \frac{M_\alpha t_1^\alpha}{\alpha} \sup_{s\in[0,t_1)} |f(s,u_n(s)) - f(s,u(s))|$$

and assuming continuity of the f, $||F_2u_n - F_2u|| \to 0$ as $n \to \infty$. Therefore F_2 is continuous over the interval $[0, t_1)$.

To prove, $\{F_2u(t), u \in B_{r_0}\}$ is relatively compact it is sufficient to show that the family of functions $\{F_2u, u \in B_{r_0}\}$ is uniformly bounded, equicontinuous and for any $t \in [0, t_1)$, $\{F_2u(t), u \in B_{r_0}\}$ is relatively compact in $\mathbb{U}$.
Clearly, for any $u \in B_{r_0}$, $||F_2u|| \leq r_0$, which means that the family $\{F_2u(t), u \in B_{r_0}\}$ is uniformly bounded.
For any $u \in B_{r_0}$ and $0 \leq \tau_1 < \tau_2 < t_1$,

$$\begin{aligned}|F_2u(\tau_2) - F_2(\tau_1)| &= \left|\int_0^{\tau_2} (\tau_2-s)^{\alpha-1}T_\alpha(\tau_2-s)f(s,u(s))ds - \int_0^{\tau_1} (\tau_1-s)^{\alpha-1}T_\alpha(\tau_1-s)f(s,u(s))ds\right|\\ &\leq \left|\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}T_\alpha(\tau_2-s)f(s,u(s))ds\right| + \left|\int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]\right.\\ &\quad \left. T_\alpha(\tau_2-s)f(s,u(s))ds\right| + \left|\int_0^{\tau_1} (\tau_1-s)^{\alpha-1}\big[T_\alpha(\tau_2-s) - T_\alpha(\tau_1-s)\big]f(s,u(s))ds\right|\\ &\leq I_1 + I_2 + I_3,\end{aligned}$$

Where,

$$\begin{aligned}I_1 &= \left|\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}T_\alpha(\tau_2-s)f(s,u(s))ds\right| \leq \int_{\tau_1}^{\tau_2} |(\tau_2-s)^{\alpha-1}T_\alpha(\tau_2-s)f(s,u(s))|ds\\ &\leq \frac{M_\alpha M_1}{(1+b)^{1-\beta}}(\tau_2-\tau_1)^{(1+b)(1-\beta)}||u|| \quad \text{(Applying inequality (4.4) over interval } [\tau_1,\tau_2]),\end{aligned}$$

$$\begin{aligned}I_2 &= \left|\int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]T_\alpha(\tau_2-s)f(s,u(s))ds\right|\\ &\leq M_\alpha \int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]|f(s,u(s))|ds \leq M_\alpha M_1\left(\int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]^{\frac{1}{1-\beta}}ds\right)^{1-\beta}||u||\end{aligned}$$

Applying Holder inequality

$$\begin{aligned}I_2 &\leq M_\alpha M_1\left(\int_0^{\tau_1} [(\tau_2-s)^b - (\tau_1-s)]ds\right)^{1-\beta}||u||\\ &\leq \frac{M_\alpha M_1}{(1+b)^{1-\beta}}\left(\tau_1^{1+b} - \tau_2^{1+b} + (\tau_2-\tau_1)\right)^{1-\beta}||u|| \leq \frac{MM_1}{\Gamma(\alpha)(1+b)^{1-\beta}}(\tau_2-\tau_1)^{(1+b)(1-\beta)}||u||\end{aligned}$$

and finally,

$$\begin{aligned}I_3 &= \left|\int_0^{\tau_1} (\tau_1-s)^{\alpha-1}\big[T_\alpha(\tau_2-s) - T_\alpha(\tau_1-s)\big]f(s,u(s))ds\right|\\ &\leq \int_0^{\tau_1} |(\tau_1-s)^{\alpha-1}T_\alpha(\tau_2-s) - T_\alpha(\tau_1-s)f(s,u(s))|ds\end{aligned}$$

$$\leq \int_0^{\tau_1} |(\tau_1 - s)^{\alpha-1} f(s, u(s) \sup_{s\in[\tau_1,\tau_2]} |T_\alpha(\tau_2 - s) - T_\alpha(\tau_1 - s)|ds||u||$$
$$\leq \frac{M_1}{(1+b)^{1-\beta}} t^{(1+b)(1-\beta)} \sup_{s\in[\tau_1,\tau_2]} |T_\alpha(\tau_2 - s) - T_\alpha(\tau_1 - s)|||u||, \text{ (Applying Holder's inequality).}$$

The integrals I_1 and I_2 are vanishes if $\tau_1 \to \tau_2$ as they contain term $(\tau_2 - \tau_1)$. Assuming (A1), the integral I_3 also vanishes. Therefore, $|F_2u(\tau_2) - F_2(\tau_1)|$ also vanishes. Hence, the family $\{F_2u, u \in B_{r_0}\}$ is equicontinuous.
Now we show that the family $X(t) = \{F_2u(t), u \in B_{r_0}\}$ for all $t \in [0, t_1)$ is relatively compact. It is obvious that $X(0)$ is relatively compact.
Let $t_0 \in [0, t_1)$ be fixed and for each $\epsilon \in [0, t_1)$, define an operator F_ϵ on B_{r_0} as

$$F_\epsilon u(t) = \int_0^{t-\epsilon} (t-s)^{\alpha-1} T_\alpha(t-s) f(t, u(s))ds.$$

Compactness of the operator $VT_\alpha t)$ leads to relative compactness of the set $X_\epsilon(t) = F_\epsilon u(t), u \in B_{r_0}$ in $\mathcal{U}$. Moreover,

$$|F_2u(t) - F_\epsilon u(t)| = \left| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(t, u(s))ds - \int_0^{t-\epsilon} (t-s)^{\alpha-1} T_\alpha(t-s) f(t, u(s))ds \right|$$
$$\leq \int_\epsilon^t |(t-s)^{\alpha-1} T_\alpha(t-s) f(t, u(s))|ds$$
$$\leq \frac{M_\alpha M_1}{(1+b)^{1-\beta}} (t-\epsilon)^{(1+b)(1-\beta)} \quad \text{(Applying inequality (4.4)).}$$

Thus, $X(t)$ is relatively compact as it is very closed to relatively compact set $X_\epsilon(t)$. Therefore, by Ascoli-Arzela theorem the operator F_2 is completely continuous on B_{r_0}. Hence, by Krasnoselskii's fixed point theorem $F_1 + F_2$ has fixed point on B_{r_0} which is mild solution of the equation (4.1) over the interval $[0, t_1)$.
Over the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as,

$$F_1u(t) = g_k(t, u(t))$$
$$F_2u(t) = 0$$

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$$F_2u(t) = 0$$

Assuming, (B3) the operators F_1 and F_2 satisfies all the requirements of the Kranoselskii's fixed point theorem. Therefore the mild solution of the equation (4.1) over the interval $[t_k, s_k)$ exist. Moreover, it also satisfies the hypotheses of the Banach fixed point theorem the value of g_k at s_k is well define.
Over the interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as,

$$F_1u(t) = T(t - s_k) g_k(s_k, u(s_k))$$
$$F_2u(t) = \int_{s_k}^t (t-s)^{\alpha-1} T_\alpha(t-s) f(t, u(s))ds$$

then, $u(t)$ is mild solution of the semilinear fractional evolution equation if and only if the operator equation $u = F_1u + F_2u$ has solution for $u \in B_r$ for some r. Therefore the existence of a mild solution of (4.1) over the interval $[s_k, t_{k+1})$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}.
Using similar arguments for interval $[0, t_1)$ and by Krasnoselskii's fixed point theorem $F_1 + F_2$ has fixed point on B_{r_0} which is mild solution of the equation (4.1) over the interval $[s_k, t_{k+1})$. This completes the proof of the theorem. □

Example 4.1. *Fractional evolution equation non-local conditions:*

$$\begin{aligned} {}^cD^{1/2}u(t,x) &= u_{xx}(t,x) + \frac{1}{50}e^{-u(s,x)}, \quad t \in [0,\frac{1}{3}) \cup [\frac{2}{3},1] \\ u(t,x) &= \frac{u(t,x)}{10(1+u(t,x))}, \quad t \in [\frac{1}{3},\frac{2}{3}) \end{aligned} \tag{4.5}$$

over the interval $[0,1]$ *with initial condition* $u(0,x) = u_0(x) + \sum_{i=1}^{2} \frac{1}{3^i}u(1/i,x)$ *and boundary condition* $u(t,0) = u(t,1) = 0$.
The domain of the operator $Au = -\frac{\partial^2 u}{\partial x^2}$ *is* $\mathcal{D}(A) = \{z \in L^2[0,1]/z$ *is twice differentiable and* $z(0) = z(1) = 0\}$.
The function $u(t)$ *is mild solution of* (4.1) *in the interval* $[0,1]$ *if* $u(t)$ *satisfies:*

$$u(t) = \begin{cases} T(t)[u_0 + h(u)] + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\{\frac{1}{50}e^{-u(s,x)}\}ds, & t \in [0,\frac{1}{3}) \\ \frac{u(t,x)}{10(1+u(t,x))}, & t \in [\frac{1}{3},\frac{2}{3}) \\ T(t-2/3)\frac{u(2/3,x)}{10(1+u(2/3,x))} + \int_{2/3}^t (t-s)^{\alpha-1}T_\alpha(t-s)\{\frac{1}{50}e^{-u(s,x)}\}ds, & t \in [\frac{2}{3},1] \end{cases} \tag{4.6}$$

Here,

$$T(t)z = \sum_{n=1}^{\infty} E_\alpha(-n^2t^\alpha) < z, \phi_n > \phi_n \text{ and } T_\alpha(t)z = \sum_{n=1}^{\infty} E_{\alpha,\alpha}(-n^2t^\alpha) < z, \phi_n > \phi_n$$

are the generators of the linear operator A. $\phi_n(x)$ *are orthogonal Fourier basis functions in* $L^2[0,1]$.
Clearly, the operators in the equations satisfies all the required conditions of theorem. Therefore the equation (4.5) *has mild solution over the interval* $[0,1]$.

5. CONCLUSION

Existence of mild solution of non-instantaneous impulsive semi-linear evolution equation with classical and non-local conditions over the general Banach space is established in this paper. The result of classical evolution equation is obtained through the the concept of generators and general Banach contraction theorem, while the non-local evolution equation is obtained through concept of generators and Krasnoselskii's fixed point theorem.

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