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Isolated Connected Domination In Graphs

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Abstract: An isolated connected dominating set (ICD-set) S of a connected graph G is a dominating set $S \subseteq V(G)$ such that $< S >$ is a union of a connected graph(nonisolated graph) and some(at least one) isolated vertices. An isolated connected domination number of G, denoted by $\gamma_{ic}(G)$, is the minimum cardinality of an isolated connected dominating set of G. In this paper, we study some properties of ICD and we give isolated connected domination number of some families of graphs.
AMS Subject Classification: 05C69

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1. INTRODUCTION

Throughout this paper, we consider only finite, simple and undirected graphs. The set of vertices and edges of a graph $G(p,q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$. For graph theoretic terminology, we follow [7].
For $v \in V(G)$, the open neighborhood of v is $N_G(v) = \{u \in V: uv \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $deg_G(v) = |N_G(v)|$. The minimum and maximum degree of G is defined by $\delta(G) = \min_{v \in V(G)} \{deg(v)\}$ and $\Delta(G) = \max_{v \in V(G)} \{deg(v)\}$ respectively. A vertex of degree zero is called an isolated vertex.

A subset S of vertices of a graph G is a dominating set of G if every vertex in $V(G) - S$ has a neighbor in S. The minimum cardinality of a dominating set of G as called the domination number and is denoted by $\gamma(G)$. A dominating set S of a connected graph G is a connected dominating set if $< S >$ is a connected subgraph of G. The minimum cardinality of a connected dominating set of G is called the connected domination number and is denoted by $\gamma_c(G)$.
In 2016, Hameed and Balamurugan [11] introduced the concept of isolate domination in graphs. A dominating set S of a graph G is said to be an isolate dominating set if $< S >$ has at least one isolated vertex [11]. An isolate dominating set S is said to be minimal if no proper subset of S is an isolate dominating set. The minimum and maximum cardinality of a minimal isolate dominating set of G are called the isolate domination number $\gamma_0(G)$ and the upper isolate domination number $\Gamma_0(G)$ respectively.
By using the definition of connected dominating set and isolate domination, we introduced the concept of isolated connected dominating set in graphs. An isolated connected dominating set (ICD-set) S of a graph G is a dominating set $S \subseteq V(G)$ such that $< S >$ is a union of a connected graph and some(at least one) isolated vertices. An isolated connected domination

number of G, denoted by $\gamma_{ic}(G)$, is the minimum cardinality of an isolated connected dominating set of G.
we study some properties of ICD-set and we give isolated connected domination number of some families of graphs.

2. MAIN RESULTS

In this section we study some important properties of ICD sets. From the definition of ICD it must include atleast two adjacent vertices and at least one isolated vertex and so we have the following result.

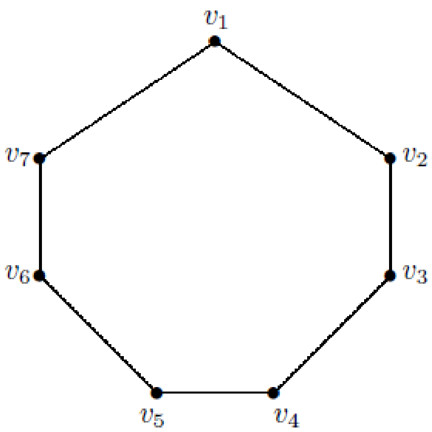
Lemma 1.*If a graph G admits ICD-set, then* $diam(G) \geq 3$.

The next result gives the basic relationship between connected domination number and ICD number.

Lemma 2. *If a graph G admits ICD-set, then* $\gamma(G) \leq \gamma_{ic}(G)$.

Proof. Since every ICD-set is a dominating set, we have $\gamma(G) \leq \gamma_{ic}(G)$.

Remark 3 . *There is no relationship between γ_c and γ_{ic}. For example, consider the following graphs G_1 and G_2. The set $\{v_1, v_2, v_3, v_4, v_5\}$ is minimum connected dominating set of G_1 and $\gamma_c(G_1) = 5$. Also the set $\{v_1, v_4, v_5\}$ is minimum ICD-set of G_1 and $\gamma_{ic}(G_1) = 3$.*
For the graph G_2, the set $\{w_3, w_6\}$ is minimum connected dominating set and $\gamma_c(G_2) = 2$ but $\gamma_{ic}(G_2) = 4$(since $\{w_2, w_3, w_4, w_5\}$ is minimum ICD-set).



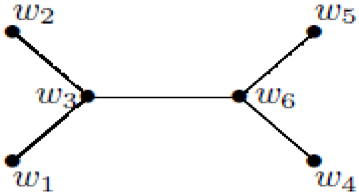


Figure 1: G_1Figure 2: G_2

Theorem 4 . *Let $n \geq 2$ be an integer and let G be a disconnected graph with n components G_1, G_2, ...G_n. Then G admits ICD-set if and only if at least $n-1$ components of G has a full vertex(full vertex with respect to corresponding component).*
For this case, when all the components have full vertex, the isolated connected domination number is given by $\gamma_{ic}(G) = n + 1$.

Proof. Suppose G admits ICD-set. let S be a minimum ICD-set of G. By the definition of S and G, $< S >$ must have at least n components and so atleast $n-1$ isolated vertices. This means that $|S \cap V(G_i)| = 1$ for at least $n-1$ components of G. That is, all those $n-1$ components have a full vertex.
Conversely suppose at least $n-1$ components of G has a full vertex, say $G_1, G_2, \dots, G_{n-1}$. Since every connected graph admits connected dominating set, the set S consist of one full vertex from each G_i for $1 \leq i \leq n-1$ and a minimum connected dominating set of G_n is a ICD-set of G.
Suppose the component G_n also has a full vertex. Note that G mus have at least one component with more than one vertex (otherwise G is a null graph, which does not admit ICD-set), let it be G_1. Let a be a full vertex of G_1 and b be any vertex adjacent to a. Then the set $\{a, b\}$ together with a set consist of a full vertex from each other component is a minimum ICD-set of G with $n+1$ vertices.

Lemma 5. *For any connected graph H with $|V(H)| \geq 3$, there is a vertex $a \in V(H)$ such that $H - \{a\}$ is connected.*

Proof. Let a, b be two vertices such that $d(a,b) = diam(H)$ and P be a longest path between a and b. Consider the graph $H - \{a\}$. Let $u \in H - \{a\}$.
case 1: If $u \in P$, then u and b are connected in $H - \{a\}$ through the path $P - \{a\}$.
case 2: If $u \notin P$, then there exists a path Q of length less than or equal to $d(a,b)$ in G which does not pass through a. Thus Q is also a path in $H - \{a\}$, which means that u and b are connected in $H - \{a\}$. Thus $H - \{a\}$ is a connected graph. Note that $H - \{a\}$ is not isomorphic to K_1(since $|V(H)| \geq 3$).

Lemma 6. *Let G be a connected graph of order $n \geq 3$. Then*
(a) $\gamma_c(G) \neq n$. (b) $\gamma_{ic}(G) \neq n$ if G admits ICD-set.

Proof. Let G be a connected graph of order $n \geq 2$.
(a) By Lemma 5, there is a vertex $a \in V(G)$ such that $V(G) - \{a\}$ is connected, which is a connected dominating set of G with $n-1$ elements.
(b) Let S be a minimum ICD-set of G. Suppose $|S| = n$. Then $< S >= G$, which has no isolated vertex, a contradiction. Thus $\gamma_{ic}(G) \neq n$

Lemma 7. *Let G be a graph of order $n \geq 4$. If G has no isolated vertex in it and G admits ICD-set with $\gamma_{ic}(G) = n-1$, then G must be connected or G is a union of two K_2.*

Proof. Let G be a graph having no isolated vertex and G admits ICD-set. Suppose G is not connected. Let k be the number of components of G.
Suppose $k \geq 3$, say $H_1, H_2, \dots, H_k$. Since G has no isolated vertex, each component has at least two vertices. By Theorem 4, at least $k-1$ components have full vertex(component wise) in it. Let H_k be the only component which may not have a full vertex. Then any connected dominating set together with these $k-1$ full vertices forms a ICD-set of G with less than $n-1$ elements, a contradiction.
Let H_1 and H_2 be the two components of G.

Note that $|V(H_1)|$, $|V(H_2)| \geq 2$. By Theorem 4, either H_1 or H_2 must have a full vertex. With out loss of generality, let x be a full vertex in H_1.
Case 1: Suppose H_1 and H_2 are not isomorphic to K_2.
By Lemma 6, $\gamma_c(H_2) \neq |V(H_2)|$. Since $|V(H_2)| \geq 3$, by Lemma 5, there is a vertex $a \in V(H_2)$ such that $V(H_2) - \{a\}$ is connected, which is a connected dominating set of H_2 with $|V(H_2)| - 1$ elements. In this case, the set $\{x\} \cup (H_2 - \{a\}) \cup (G - (V(H_1) \cup V(H_2)))$ is an ICD set of G with less than $n - 1$ elements, a contradiction.
Case 2: Suppose $H_1 \approx K_2$ and H_2 is not isomorphic to K_2.
As proved in Case 1, a full vertex of H_1 together with a minimum connected dominating set of H_2 forms a minimum ICD-set of G with less than $n - 1$. elements.
Case 3: Suppose $H_2 \approx K_2$ and H_1 is not isomorphic to K_2.

As proved in Case 1, a full vertex of H_2 together with a minimum connected dominating set of H_1 forms a minimum ICD-set of G with less than $n - 1$. elements.
From all the above cases, it is easy to conclude that G is a union of two K_2.

Lemma 8 .*Let G be a disconnected graph of order $n \geq 2$. Then $\gamma_{ic}(G) = n - 1$ if G is a union of some isolated vertices(at least one isolated) and a graph H, where $H = K_3$ or P_3; (or) G is a union of two copies of K_2.*

Proof. Case 1: Suppose G is a union of some isolated vertices(at least one isolated) and a graph H, where $H = K_3$ or P_3.
In this case all the isolated vertices of G together with any two adjacent vertices of H forms a minimum ICD-set with $n - 1$ vertices.
Case 2: Suppose G is a union of two copies of K_2.
In this case, G has four vertices and any three vertices of G forms a minimum ICD-set with $n - 1 = 3$ vertices.

Theorem 9. *For given integer $k \geq 1$, there exists a graph G such that $\gamma_c(G) = \gamma_{ic}(G) = k$.*

Proof. Let G be a graph obtaining from a path P_k ($V(P_k) = \{a_1, a_2, \dots, a_k\}$) by adding one pendent vertex b_i at each a_i such that $a_i b_i \in E(G)$. Since every pendent or corresponding stem must be in every dominating set, either a_i or b_i must be in every dominating set of G and so $\gamma(G) \geq k$. Since $\gamma(G) \leq \gamma_c(G)$ and $V(P_k)$ is a connected dominating set of G, it follows that $\gamma_c(G) = k$. Since $\gamma(G) \leq \gamma_{ic}(G)$ and $(V(P_k) - a_1) \cup b_1$ is a isolated connected dominating set of G, it follows that $\gamma_c(G) = k$.

Lemma 10 . *For an integer $n \geq 5$, the path P_n admits ICD-set with ICD number $\gamma_{ic}(P_n) = \lceil\frac{n-4}{3}\rceil + 2$.*

Proof. Let $V(P_n) = \{a_i / 1 \leq i \leq n\}$ and $E(P_n) = \{a_i a_{i+1} / 1 \leq i \leq n - 1\}$.
Let S be any ICD-set P_n. Then $< S >$ must have at least two adjacent vertices, say a, b. Note

that a and b can dominate a maximum of 4 vertices(including a and b) of P_n. Also every other vertex u of S can dominate a maximum of 3 different vertices(including u). Thus to dominate the remaining undominated $n-4$ vertices of P_n, S must have $\lceil\frac{n-4}{3}\rceil$ vertices excluding a and b.
Thus $|S| \geq \lceil\frac{n-4}{3}\rceil + 2$ and so $\gamma_{ic}(P_n) \geq \lceil\frac{n-4}{3}\rceil + 2$.

Case 1: Suppose $n = 3k+1$ for some $k \geq 2$.
Then $\{a_2, a_3\} \cup \{a_{3i}: i = 2,3, \dots, k\}$ is a ICD-set with $k+1$ elements and $\lceil\frac{n-4}{3}\rceil + 2 = \lceil\frac{(3k+1)-4}{3}\rceil + 2 = k-1) + 2 = k+1$.

Case 2: Suppose $n = 3k+2$ for some $k \geq 1$.
Then $\{a_2, a_3\} \cup \{a_{3i+2}: i = 1,2, \dots, k\}$ is a ICD-set with $k+2$ elements and $\lceil\frac{n-4}{3}\rceil + 2 = \lceil\frac{(3k+2)-4}{3}\rceil + 2 = k+2$.

Case 3: Suppose $n = 3k$ for some $k \geq 2$.
Then $\{a_2, a_3\} \cup \{a_{3i+2}: i = 1,2, \dots, k-1\}$ is a ICD-set with $k+1$ elements and $\lceil\frac{n-4}{3}\rceil + 2 = \lceil\frac{3k-4}{3}\rceil + 2 = (k-1) + 2 = k+1$. Thus in all the cases, these exists a ICD-set of P_n with $\lceil\frac{n-4}{3}\rceil + 2$ elements.

Remark 11. *Since $diam(P_2), diam(P_3) \leq 2$, the paths P_2 and P_3 does not admit ICD-set. The set $\{a_1, a_2, a_4\}$ is a minimum ICD with 3 elements and so $\gamma_{ic}(P_4) = 3$.*
By taking, $V(C_n) = \{a_i / 1 \leq i \leq n\}$ and $E(C_n) = \{a_n a_1\} \cup \{a_i a_{i+1} / 1 \leq i \leq n-1\}$, as in the proof of above lemma, we can prove the following.

Lemma 12 . *For an integer $n \geq 5$, the path C_n admits ICD-set with ICD number $\gamma_{ic}(C_n) = \lceil\frac{n-4}{3}\rceil + 2$.*

Remark 13. *Since $diam(C_2), diam(C_3) \leq 2$, the cycle graphs C_3 and C_4 does not admit ICD-set. The set $\{a_1, a_2, a_4\}$ is a minimum ICD with 3 elements and so $\gamma_{ic}(P_4) = 3$.*

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