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# Further Results On F-average Eccentric Graphs

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***Abstract:* The F-average eccentric graph $AE_F(G)$ of a graph $G$ has the vertex set as in $G$ and any two vertices $u$ and $v$ are adjacent in $AE_F(G)$ if either they are at a distance $\left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$ while $G$ is connected or they belong to different components while $G$ is disconnected. A graph $G$ is called a F-average eccentric graph if $AE_F(H) \cong G$ for some graph $H$. In this paper, we find some sufficient conditions for a disconnected graph to be or not to be a F-average eccentric graph.**

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***Keywords:* F-average eccentric vertex, F-average eccentric graph.**

## 1. INTRODUCTION

Throughout this paper, a graph means a non trivial simple graph. For other graph theoretic notation and terminology, we follow [8,9]. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. $d(v)$ denotes the degree of a vertex $v \in V(G)$, the order of $G$ is $|V(G)|$ and the size is $|E(G)|$. The distance $d(u,v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The radius $r(G)$ of $G$ is the minimum eccentricity among the eccentricities of the vertices of $G$ and the diameter $d(G)$ of $G$ is the maximum eccentricity among the eccentricities of the vertices of $G$. Splitting graph $S(G)$ of a graph $G$ was introduced by Sampath Kumar and Walikar [6]. For each vertex $v$ of a graph $G$, take a new vertex $v'$ and join $v'$ to all the vertices of $G$ adjacent to $v$. The graph $S(G)$ thus obtained is called the splitting graph of $G$. A vertex $v$ is called an eccentric vertex of a vertex $u$ if $d(u,v) = e(u)$. A vertex $v$ of $G$ is called an eccentric vertex of $G$ if it is the eccentric vertex of some vertex of $G$. Let $S_i(G)$ denote a subset of the vertex set of $G$ such that $e(u) = i$ for all $u \in V(G)$. The concept of antipodal graph was initially introduced by Singleton [1] and was further expanded by Aravamuthan and Rajendran [3,4]. The antipodal graph of a graph $G$, denoted by $A(G)$, is the graph on the same vertices as of $G$, two vertices being adjacent if the distance between them is equal to the diameter of $G$. A graph is said to be antipodal if it is the antipodal $A(H)$ of some graph $H$. The concept of eccentric graph was introduced by Akiyama et al. [2]. The eccentric graph based on $G$ is denoted by $G_e$ whose vertex set is $V(G)$ and two vertices $u$ and $v$ are adjacent in $G_e$ if $d(u,v) = min\{e(u), e(v)\}$. The concept of radial graph was introduced by Kathiresan and Marimuthu [5]. The radial graph $R(G)$ based on $G$ has the vertex set as in $G$ and two vertices are adjacent if the distance between them is equal to the radius of $G$ while $G$ is connected. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of $G$. A graph $G$ is called a radial graph if $R(H) = G$ for some graph $H$. Sathiyanandham and Arockiaraj introduced a new graph, called $F$-average eccentric graph [7]. Two vertices $u$ and $v$ of a graph are said to be $F$-average eccentric to each other if $d(u,v) = \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$. The $F$-average eccentric graph of a graph $G$, denoted by $AE_F(G)$, has the vertex set as in $G$ and any two vertices $u$ and $v$ are adjacent in $AE_F(G)$ if either they are at a distance $d(u,v) = \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$ while $G$ is connected or they belong to different components while $G$ is disconnected. A graph $G$ is called a $F$-average eccentric graph if $AE_F(H) \cong G$ for some graph $H$. In this paper, we find some sufficient conditions for a disconnected graph to be or not to be a $F$-average eccentric graph.

Let $F_{22}$ be the set of all connected graphs $G$ for which $r(G) = d(G) = 2$.

**Theorem A[7]** Let $G$ be a graph on $n$ vertices. Then a vertex is a full degree vertex in $AE_F(G)$ if and only if either it is an isolated vertex or a full degree vertex or a non full degree vertex adjacent to the full degree vertices only in $G$.

**Theorem B[7]** For any graph $G \in F_{22}, AE_F(G) = \overline{G}$.

## 2. RESULTS ON $F$-AVERAGE ECCENTRIC GRAPHS

**Proposition 2.1.** If $G$ is a disconnected graph with no isolated vertex, then $G$ is a $F$-average eccentric graph.

**Proof.** By hypothesis, $\overline{G} \in F_{22}$ and by Theorem B, $AE_F(\overline{G}) \cong \overline{\overline{G}} = G$ . □

**Theorem 2.2.** If $G$ is a disconnected graph having a component of the form $K_{r_1,r_2,\ldots,r_n} - E(K_n)$ where $r_1, r_2, \ldots, r_n$ are positive integers, then $G$ is a $F$- average eccentric graph.

**Proof.** In $K_{r_1,r_2,\ldots,r_n}$, let $V_i = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_{r_i}^{(i)}\}$ be the $i^{th}$ partition of $K_{r_1,r_2,\ldots,r_n}$, $1 \le i \le n$. Let $V(K_n) = \{u_i \in V_i : i = 1,2,\ldots,n\}$. By graph symmetry, assume that $u_i = v_1^{(i)}$ for each $i = 1,2,\ldots,n$ and $E(K_n) = \{v_1^{(i)} v_1^{(k)} : i \ne k, 1 \le i,k \le n\}$. Construct $H$ from $G$ as follows: $H_1, H_2$ are two partitions of $K_{r_1,r_2,\ldots,r_n}$ and $V(H_3) = V(G) - V(K_{r_1,r_2,\ldots,r_n} - E(K_n))$ where $V(H_1) = \{v_j^{(i)} \in V_i : 2 \le j \le r_i, 1 \le i \le n\}$ and $V(H_2) = \{v_1^{(i)} \in V_i : 1 \le i \le n\}$ . $E(H) = \{v_j^{(i)} v_1^{(i)} : 2 \le j \le r_i, 1 \le i \le n\} \cup \{v_1^{(i)} w : w \in V(H_3), 1 \le i \le n\} \cup$ $E(\overline{G - (K_{r_1,r_2,\ldots,r_n} - E(K_n))})$. For $2 \le j \le r_i$ and $1 \le i \le n$, $e(v_j^{(i)}) = 4$, $e(v_1^{(i)}) = 3$ and the eccentricities of the remaining vertices of $H$ are 2. Also $d_H(v_j^{(i)}, v_{j'}^{(k)}) = 4$ , $d_H(v_j^{(i)}, v_1^{(k)}) = 3$, $d_H(v_j^{(i)}, v_1^{(i)}) = 1$ and $d_H(v_1^{(i)}, v_1^{(k)}) = 2$ for $2 \le j \le r_i$, $2 \le j' \le r_k$, $i \ne k$ and $1 \le i,k \le n$, $d_H(v_j^{(i)}, u) = 2$ and $d_H(v_1^{(i)}, u) = 1$ for $u \in V(H_3)$, $2 \le j \le r_i$ and $1 \le i \le n$, $d_H(v,w) = 2$ for every non adjacent pairs of vertices $v$ and $w$ in $V(H_3)$. This implies that $d_H(v_j^{(i)}, v_{j'}^{(k)}) = \left\lfloor \frac{e(v_j^{(i)})+e(v_{j'}^{(k)})}{2} \right\rfloor$, $d_H(v_j^{(i)}, v_1^{(k)}) = \left\lfloor \frac{e(v_j^{(i)})+e(v_1^{(k)})}{2} \right\rfloor$ for $2 \le j \le r_i$, $2 \le j' \le r_k, i \ne k$ and $1 \le i,k \le n$, $d_H(u,w) = \left\lfloor \frac{e(u)+e(w)}{2} \right\rfloor$ for every non adjacent pairs of vertices $u$ and $w$ in $V(H_3)$. Also $d_H(v_j^{(i)}, v_1^{(i)}) < \left\lfloor \frac{e(v_j^{(i)})+e(v_1^{(i)})}{2} \right\rfloor$ and $d_H(v_1^{(i)}, v_1^{(k)}) < \left\lfloor \frac{e(v_1^{(i)})+e(v_1^{(k)})}{2} \right\rfloor$ for $2 \le j \le r_i, i \ne k$ and $1 \le i,k \le n$, $d_H(v_j^{(i)}, u) < \left\lfloor \frac{e(v_j^{(i)})+e(u)}{2} \right\rfloor$ and $d_H(v_1^{(i)}, u) < \left\lfloor \frac{e(v_1^{(i)})+e(u)}{2} \right\rfloor$ for $u \in V(H_3)$, $2 \le j \le r_i$ and $1 \le i \le n$, $d_H(u,w) < \left\lfloor \frac{e(u)+e(w)}{2} \right\rfloor$ for every adjacent pairs of vertices $u$ and $w$ in $V(H_3)$. Hence $E(AE_F(H))=\{v_j^{(i)} v_{j'}^{(k)}, v_j^{(i)} v_1^{(k)}: 2 \le j \le r_i, 2 \le j' \le r_k, i \ne k, 1 \le i,k \le n\} \cup E(\overline{H_3})=E(G)$. Thus $G$ is a $F$ - average eccentric graph. □

**Corollary 2.3.** If $G$ is a disconnected graph having a component of the form $K_{m,n} - e$ where $m$ and $n$ are positive integers, then $G$ is a $F$- average eccentric graph.
**Proof.** By taking $p = 2$ in Theorem 2.2, the result follows. □

**Corollary 2.4.** If $G$ is a disconnected graph having $P_4$ as a component, then $G$ is a $F$- average eccentric graph.
**Proof.** Since $P_4 \cong K_{2,2} - e$ , by Corollary 2.3, the result follows. □

**Corollary 2.5.** If $G$ is a disconnected graph having a component of the form $S(K_m)$, $m$ being a positive integer $\ge 3$, then $G$ is a $F$- average eccentric graph.
**Proof.** By taking $n = m$ and $r_1 = r_2 = \ldots = r_m = 2$ in Theorem 2.2, the result follows. □

Let $V_i = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_{r_i}^{(i)}\}$ be the $i^{th}$ partition of $K_{r_1,r_2,\ldots,r_n}$ for $i = 1,2,\ldots,n$. By deleting all the edges between the successive $m_i^{th}$ and $m_{i+1}^{th}$ partitions of $K_{r_1,r_2,\ldots,r_n}$ in a cyclic manner, the resulting graph is denoted as $K_{r_1,r_2,\ldots,r_n}^{(m_1,m_2,\ldots,m_l)}$. That is, $K_{r_1,r_2,\ldots,r_n}^{(m_1,m_2,\ldots,m_l)} = K_{r_1,r_2,\ldots,r_n} - \{v_{j_t}^{(m_t)} v_{j_{t'}}^{(m_{t+1})}: v_{j_{t'}}^{(m_1)} = v_{j_{t'}}^{(m_{l+1})}, 1 \le j_t \le r_{m_t}, 1 \le j_{t'} \le r_{m_{t+1}}, 1 \le t,t' \le l\}$ for $1 \le m_t \le n$, $2 \le l \le n$. In particular $K_{r_1,r_2,\ldots,r_n}^{(1,2,\ldots,m)}=K_{r_1,r_2,\ldots,r_n} - \{v_j^{(t)} v_{j'}^{(t+1)}: v_{j'}^{(1)} = v_{j'}^{(m+1)}, 1 \le j \le r_t, 1 \le j' \le r_{t+1}, 1 \le t \le m\}$ for $2 \le m \le n$. Let $v_0, v_1, \ldots, v_{m-1}$ be the vertices of a complete graph $K_m$, $m \ge 3$ and $w_i$, $0 \le i \le m-1$, be the duplicating vertices of $v_i, 0 \le i \le m-1$ respectively. Suppose that $v_{m+i} = v_i, 0 \le i \le m-1$. Then the graph $S(K_m) - \{v_i v_{i-1}, v_i v_{i+1}: 0 \le i \le n\}$ is denoted by $S'(K_m)$. That is, $S'(K_m) = K_{2,2,\ldots,2}^{(1,2,\ldots,m)}$

**Theorem 2.6.** If $G$ is a disconnected graph having a component of the form $K_{r_1,r_2,\ldots,r_n}^{(m_1,m_2,\ldots,m_l)}$ for $1 \le m_t, l \le n$, $1 \le t \le l$, $n \ge 4$ and at least one pair of positive numbers in $\{m_1, m_2, \ldots, m_l\}$ is not equal, then $G$ is a $F$- average eccentric graph.
**Proof.** In $K_{r_1,r_2,\ldots,r_n}^{(m_1,m_2,\ldots,m_l)}$, $V_i = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_{r_i}^{(i)}\}$ is the $i^{th}$ partition of $K_{r_1,r_2,\ldots,r_n}$ for $1 \le i \le n$. Construct $H$ from $G$ as follows: Let $H_1$ and $H_2$ be two partitions of $K_{r_1,r_2,\ldots,r_n}^{(m_1,m_2,\ldots,m_l)}$ where $V(H_1) = \{v_j^{(i)} \in V_i: 2 \le j \le r_i, 1 \le i \le n\}$ and $V(H_2) = \{v_1^{(i)} \in V_i: 1 \le i \le n\}$. Let

$V(H_3) = V(G) - V(K_{r_1,r_2,\dots,r_n}^{(m_1,m_2,\dots,m_l)})$ and $E(H) = \{v_j^{(i)}v_1^{(i)}, \overline{v_1^{(m_t)}v_1^{(m_{t+1})}: m_t = m_{t+l}}, 2 \le j \le r_i, 1 \le i, m_t \le n, 1 \le t \le l\} \cup \{v_1^{(i)}w: w \in V(H_3), 1 \le i \le n\} \cup E(\overline{G - K_{r_1,r_2,\dots,r_n}^{(m_1,m_2,\dots,m_l)}})$. For $2 \le j \le r_i$ and $1 \le i \le n$, $e(v_j^{(i)}) = 4$, $e(v_1^{(i)}) = 3$ and the eccentricities of the remaining vertices of $H$ are 2. Also $d_H(v_j^{(m_t)}, v_{j'}^{(k)}) = 4$, $d_H(v_{j_1}^{(i)}, v_{j'_1}^{(i')}) = 4$, $d_H(v_j^{(m_t)}, v_1^{(k)}) = 3$, $d_H(v_{j_1}^{(i)}, v_1^{(i')}) = 3$, $d_H(v_j^{(m_t)}, v_{j''}^{(m_{t+1})}) = 3$, $d_H(v_{j_2}^{(s)}, v_{j'_2}^{(s)}) = 2$, $d_H(v_j^{(m_t)}, v_1^{(m_{t+1})}) = 2, d_H(v_j^{(s)}, v_1^{(s)}) = 1$, $d_H(v_1^{(m_t)}, v_1^{(m_{t+1})}) = 1$ and $d_H(v_1^{(m_t)}, v_1^{(k)}) = 2$ for $2 \le j \le r_{m_t}$, $2 \le j' \le r_k$, $2 \le j'' \le r_{m_{t+1}}$, $2 \le j_1 \le r_i$, $2 \le j'_1 \le r_{i'}$, $j_2 \ne j'_2$, $2 \le j_2, j'_2 \le r_s$, $m_t = m_{l+t}$, $m_0 = m_l$, $k \ne m_{t-1}, m_t, m_{t+1}$; $1 \le t \le l$, $i \ne i'$, $i \ne m_t \ne i'$ and $1 \le i, i', k, s, m_t \le n$, $d_H(v_{j_2}^{(s)}, u) = 2$ and $d_H(v_1^{(s)}, u) = 1$ for $u \in V(H_3)$, $2 \le j_2 \le r_s$ and $1 \le s \le n$, $d_H(v, w) = 2$ for every non adjacent pairs of vertices $v$ and $w$ in $V(H_3)$. This implies that $d_H(v_j^{(m_t)}, v_{j'}^{(k)}) = 4 = \left\lfloor \frac{e(v_j^{(m_t)}) + e(v_{j'}^{(k)})}{2} \right\rfloor$, $d_H(v_{j_1}^{(i)}, v_{j'_1}^{(i')}) = 4 = \left\lfloor \frac{e(v_j^{(i)}) + e(v_{j'}^{(i')})}{2} \right\rfloor$, $d_H(v_j^{(m_t)}, v_1^{(k)}) = 3 = \left\lfloor \frac{e(v_j^{(m_t)}) + e(v_1^{(k)})}{2} \right\rfloor$, $d_H(v_{j_1}^{(i)}, v_1^{(i')}) = 3 = \left\lfloor \frac{e(v_j^{(i)}) + e(v_1^{(i')})}{2} \right\rfloor$ for $2 \le j \le r_{m_t}$, $2 \le j' \le r_k$, $2 \le j_1 \le r_i$, $2 \le j'_1 \le r_{i'}$, $m_t = m_{l+t}$, $m_0 = m_l$, $k \ne m_{t-1}, m_t, m_{t+1}$; $1 \le t \le l$, $i \ne i'$, $i \ne m_t \ne i'$ and $1 \le i, i', k, m_t \le n$, $d_H(v, w) = 2 = \left\lfloor \frac{e(v) + e(w)}{2} \right\rfloor$ for every non adjacent pairs of vertices $u$ and $w$ in $V(H_3)$. Also $d_H(v_j^{(m_t)}, v_{j''}^{(m_{t+1})}) = 3 < \left\lfloor \frac{e(v_j^{(m_t)}) + e(v_{j''}^{(m_{t+1})})}{2} \right\rfloor$, $d_H(v_j^{(m_t)}, v_1^{(m_{t+1})}) = 2 < \left\lfloor \frac{e(v_j^{(m_t)}) + e(v_1^{(m_{t+1})})}{2} \right\rfloor$, $d_H(v_{j_2}^{(s)}, v_1^{(s)}) = 1 < \left\lfloor \frac{e(v_{j_2}^{(s)}) + e(v_1^{(s)})}{2} \right\rfloor$, $d_H(v_1^{(m_t)}, v_1^{(k)}) = 2 < \left\lfloor \frac{e(v_1^{(m_t)}) + e(v_1^{(k)})}{2} \right\rfloor$, $d_H(v_{j_2}^{(s)}, v_{j'_2}^{(s)}) = 2 < \left\lfloor \frac{e(v_{j_2}^{(s)}) + e(v_{j'_2}^{(s)})}{2} \right\rfloor$, $d_H(v_1^{(i)}, v_1^{(i')}) = 2 < \left\lfloor \frac{e(v_1^{(i)}) + e(v_1^{(i')})}{2} \right\rfloor$ and $d_H(v_1^{(m_t)}, v_1^{(m_{t+1})}) = 1 < \left\lfloor \frac{e(v_1^{(m_t)}) + e(v_1^{(m_{t+1})})}{2} \right\rfloor$ for $2 \le j \le r_{m_t}$, $2 \le j' \le r_k$, $2 \le j'' \le r_{m_{t+1}}$, $2 \le j_1 \le r_i$, $2 \le j'_1 \le r_{i'}$, $j_2 \ne j'_2$, $2 \le j_2, j'_2 \le r_s$, $m_t = m_{l+t}$, $m_0 = m_l$, $k \ne m_{t-1}, m_t, m_{t+1}$; $1 \le t \le l$, $i \ne i'$, $i \ne m_t \ne i'$ and $1 \le i, i', k, s, m_t \le n$, $d_H(v_{j_2}^{(s)}, u) = 2 < \left\lfloor \frac{e(v_{j_2}^{(s)}) + e(u)}{2} \right\rfloor$ and $d_H(v_1^{(s)}, u) = 1 < \left\lfloor \frac{e(v_1^{(s)}) + e(u)}{2} \right\rfloor$ for $u \in V(H_3)$, $2 \le j_2 \le r_s$ and $1 \le s \le n$, $d_H(v, w) = 1 < \left\lfloor \frac{e(v) + e(w)}{2} \right\rfloor$ for every adjacent pairs of vertices $v$ and $w$ in $V(H_3)$. Hence $E(AE_F(H)) = \{v_j^{(m_t)}v_{j'}^{(k)}, v_{j_1}^{(i)}v_{j'_1}^{(i')}, v_j^{(m_t)}v_1^{(k)}, v_{j_1}^{(i)}v_1^{(i')}: 2 \le j \le r_{m_t}, 2 \le j' \le r_k, 2 \le j_1 \le r_i, 2 \le j'_1 \le r_{i'}, m_t = m_{l+t}, m_0 = m_l, k \ne m_{t-1}, m_t, m_{t+1}, 1 \le t \le l, i \ne i', i \ne m_t \ne i', 1 \le i, i', k, s, m_t \le n\} \cup E(\overline{H_3}) = E(G)$. Thus $G$ is a $F$- average eccentric graph. □

**Corollary 2.7.** If $G$ is a disconnected graph having a component of the form $K_{r_1,r_2,\dots,r_n}^{(1,2,\dots,(m-1),m)}$ for $1 \le m \le n$ and $n \ge 4$, then $G$ is a $F$-average eccentric graph.

**Corollary 2.8.** If $G$ is a disconnected graph having a component of the form $S'(K_m)$, $m$ being a positive integer $\geq 4$, then $G$ is a $F$- average eccentric graph.

**Theorem 2.9.** If $G$ is $(n-r)K_1 \cup G_1$ on $n$ vertices where $r(G_1)=1$ and $1 \leq r \leq n-1$, then $G$ is not a $F$-average eccentric graph.

**Proof.** Suppose $r(G_1)=1$ and $d(G_1)=2$. Let $u_1, u_2, \ldots, u_l$ be the full degree vertices, $u_{l+1}, u_{l+2}, \ldots, u_r$ be the non full degree vertices in $G_1$ and $u_{r+1}, u_{r+2}, \ldots, u_n$ be the isolated vertices in $G$. Suppose there exists a graph $H$ such that $AE_F(H)=G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $u_{r+1}, u_{r+2}, \ldots, u_n$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$. If $e(u_j)=d(H)$ for $r+1 \leq j \leq n$, then $u_j$ is not an isolated vertex in $AE_F(H)$, a contradiction. Therefore $1 < e(u_j) < d(H)$ for $r+1 \leq j \leq n$ and hence $e(u_i)=d(H)$ for some $i=1,2,\ldots,r$.

**Case 1.** Suppose $e(u_i)=d(H)=M$ for some $i, 1 \leq i \leq l$. Since $u_i$ is adjacent to $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_l$ in $AE_F(H)$, any one in $\{u_1, u_2, \ldots, u_{i-1}, \ u_{i+1}, \ldots, u_l\}$ is the antipodal vertex of $u_i$ in $H$.

**Case 1.1.** Suppose $d_H(u_i, u_{i+1})=M$. Let $u_i x_1 x_2 \ldots . x_{m-1} u_{i+1}$ be a diameteral path between $u_i$ and $u_{i+1}$ in $H$. Then $e(u_{i+1})=M$. Since $u_i u_k \in E(AE_F(H))$ for $i \neq k, i=1,2,\ldots,l$ and $k=1,2,\ldots,l,l+1,\ldots,r$, $x_h \not\in \{u_1, u_2, \ldots, u_r\}$ for $h=1,2,\ldots,m-1$. Therefore $x_h \in \{u_{r+1}, u_{r+2}, \ldots, u_n\}$ for $k=1,2,\ldots,m-1$. If $x_{m-1}=u_j$ and $e(u_j)=M-1$ for $r+1 \leq j \leq n$, then $u_i u_j \in E(AE_F(H))$, a contradiction. Hence $x_{m-1} \not\in V(H)$, a contradiction.

**Case 1.2.** Suppose $d_H(u_i, u_{l+1})=M$. Let $u_i y_1 y_2 \ldots . y_{m-1} u_{l+1}$ be a diameteral path between $u_i$ and $u_{l+1}$ in $H$. Then $e(u_{l+1})=M$. Since $u_i u_k \in E(AE_F(H))$ for $i \neq k, i=1,2,\ldots,l$ and $k=1,2,\ldots,l,l+1,\ldots,r$, $y_h \not\in \{u_1, u_2, \ldots, u_l\}$ for $h=1,2,\ldots,m-1$. Therefore $y_h \in V(H)-\{u_1, u_2, \ldots, u_l\}$ for $h=1,2,\ldots,m-1$. Let $y_{m-1}=u_{k_1}$ and $e(u_{k_1})=M-1$ for some $k_1=l+2, l+3, \ldots, r$. Since $u_{l+1} u_{k_1} \not\in E(AE_F(H))$, $y_{m-1} \in V(H)-\{u_1, u_2, \ldots, u_r\}$. Hence $u_i y_{m-1} \in E(AE_F(H))$, a contradiction to the fact that $y_{m-1}$ is an isolated vertex in $AE_F(H)$.

**Case 2.** Suppose $e(u_k)=d(H)=M$ for $l+1 \leq k \leq r$. Since $u_k$ is adjacent to $u_1, u_2, \ldots, u_l$ and $u_{k'}$ for some $k'=l+1, l+2, \ldots, r$, any one in $\{u_1, u_2, \ldots, u_l, u_{l+1}, \ldots, u_k, u_{k+1}, \ldots, u_r\}$ is the antipodal vertex of $u_k$ in $H$. Suppose $d_H(u_k, u_{k+1})=M$ and $u_k z_1 z_2 \ldots z_{m-1} u_{k+1}$ is a diameteral path between $u_k$ and $u_{k+1}$ in $H$. Then $e(u_{k+1})=M$. Since $u_i u_k \in E(AE_F(H))$ for $i=1,2,\ldots,l$, $z_h \not\in \{u_1, u_2, \ldots, u_l\}$ for $h=1,2,\ldots,m-1$. Therefore $z_h \in V(H)-\{u_1, u_2, \ldots, u_l\}$. Suppose $z_{m-1}=u_{k'}$ and $e(u_{k'})=M-1$ for $k'=l+1, l+2, \ldots, r$. Since $u_i u_{k'}$ and $u_i u_k \in E(AE_F(H))$ for $i=1,2,\ldots,l$, $u_i z_{h'} \not\in E(AE_F(H))$ for $h'=2,3,\ldots,m-2$, a contradiction. If $z_h \in V(H)-\{u_1, u_2, \ldots, u_r\}$, then $z_h=u_j$ for some $j=r, r+1, \ldots, n$ and hence $u_k u_j \in E(AE_F(H))$, a contradiction to the fact that $u_{k'}$ is an isolated vertex in $AE_F(H)$. Thus $AE_F(H)$ is not equal to $G$, a contradiction.

Suppose $r(G_1)=1$ and $d(G_1)=1$. Then $u_1, u_2, \ldots, u_r$ are the full degree vertices of $G_1$. Suppose there exists a graph $H$ such that $AE_F(H)=G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ must be connected. By the definition, each of $u_{r+1}, u_{r+2}, \ldots, u_n$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$. If $e(u_j)=d(H)$ for some $j$, $r+1 \leq j \leq n$, then $u_j$ is not an isolated vertex in

$AE_F(H)$, a contradiction. Therefore $1 < e(u_j) < d(H)$ for $r+1 \leq j \leq n$ and hence $e(u_i) = d(H)$ for some $i = 1,2,\ldots,r$. Let $e(u_i) = d(H) = M$. Since $u_i$ is adjacent to $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r$ in $AE_F(H)$, any one in $\{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r\}$ is the antipodal vertex of $u_i$ in $H$. Suppose $d_H(u_i, u_{i+1}) = M$ and $u_i x_1 x_2 \ldots x_{m-1} u_{i+1}$ is a diameteral path between $u_i$ and $u_{i+1}$. Then $e(u_{i+1}) = M$. Since $u_i u_k \in E(AE_F(H))$ for $i \neq k, 1 \leq i, k \leq r$, $x_h \not\in \{u_1, u_2, \ldots, u_r\}$ for $h = 1,2,\ldots,m-1$. Therefore $x_h \in \{u_{r+1}, u_{r+2}, \ldots, u_n\}$ for $h = 1,2,\ldots,m-1$. Hence $u_i x_{m-1}, x_1 u_{i+1} \in E(AE_F(H))$, a contradiction to the fact that $x_1$ and $x_{m-1}$ are isolated vertices in $AE_F(H)$. Thus $AE_F(H)$ is not equal to $G$, a contradiction. □

**Theorem 2.10.** Let $G = rK_1 \cup G^* \cup G_1 \cup G_2 \cup \ldots \cup G_p$ be a disconnected graph such that $r \geq 1$, $G^*$ is a square free component having a cycle of length $|G^*| = t \geq 3$ and each $G_i$ is also a square free component and non isomorphic to $P_4$ for $1 \leq i \leq p$. Then $G$ is not a $F$-average eccentric graph.

**Proof.** Let $v_1, v_2, \ldots, v_r$ be the isolated vertices, $v_{r+1}, v_{r+2}, \ldots, v_{r+t}$ be the vertices of $G^*$ and $v_{r+t+r_{i-1}+1}, v_{r+t+r_{i-1}+2}, \ldots, v_{r+t+r_i}$ be the vertices of the component $G_i$, $1 \leq i \leq p$ where $r_0 = 0$. Suppose there exists a graph $H$ such that $AE_F(H) = G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $v_1, v_2, \ldots, v_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$.

**Case 1.** $t \geq 5$. Since $v_{r+i}$ and $v_{r+i+1}$ are adjacent in $AE_F(H)$ for $1 \leq i \leq t-1$, there is a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $\left\lfloor \frac{e(v_{r+i}) + e(v_{r+i+1})}{2} \right\rfloor$.

**Case 1.1.** Suppose $e(v_{r+i}) + e(v_{r+i+1})$ is even and $\frac{e(v_{r+i}) + e(v_{r+i+1})}{2} = M$. If $e(v_{r+i}) \neq e(v_{r+i+1})$, then $d_H(v_{r+i}, v_{r+i+1}) < M$. So $e(v_{r+i}) = M = e(v_{r+i+1})$. Let $P_1: v_{r+i} x_1 x_2 \ldots x_{m-1} v_{r+i+1}$ be a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $M$. Suppose $v_{r+i}$ and $v_{r+k}$ are adjacent in $AE_F(H)$ for $k = i+2, i+4, i+5, \ldots, t$. Since $v_{r+i} v_{r+k} \in E(AE_F(H))$, $v_{r+k} \neq v_{r+i+1}$ and $v_{r+k} \neq x_1$. So $v_{r+k} = x_{m-1}$. Since $v_{r+i+1} v_{r+i+2} \in E(AE_F(H))$, $v_{r+i+2} \neq v_{r+i}$ and $v_{r+i+2} \neq x_{m-1}$. So $v_{r+i+2} = x_1$ and $e(v_{r+i+2}) = M - 1$. Then $v_{r+i+3}$ is a vertex in $V(H) - \{v_{r+i}, x_1, x_2, \ldots, x_{m-1}, v_{r+i+1}\}$. If $v_{r+i+3} \in \{v_1, v_2, \ldots, v_r\}$, then any one of $v_1, v_2, \ldots, v_r$ is not an isolated vertex in $AE_F(H)$ which is impossible. If $v_{r+i+3} = v_{r+t+r_{k-1}+j} \in V(G_k)$ for some $k$ and $j$, $1 \leq j \leq r_k$ and $1 \leq k \leq p$, then $v_{r+i+3} v_{r+t+r_{k-1}+j} \in E(AE_F(H))$, a contradiction. So $v_{r+i+3} \in \{v_{r+1}, v_{r+2}, \ldots, v_{r+t}\} - \{v_{r+i}, x_1, x_2, \ldots, x_{m-1}, v_{r+i+1}\}$. Then $d_H(v_{r+i}, v_{r+i+3}) = M$ and hence $v_{r+i} v_{r+i+3} \in E(AE_F(H))$. Hence $v_{r+i} v_{r+i+1} v_{r+i+2} v_{r+i+3} v_{r+i}$ is a cycle $C_4$ in $AE_F(H)$, a contradiction.

**Case 1.2.** Suppose $e(v_{r+i}) + e(v_{r+i+1})$ is odd and $\left\lfloor \frac{e(v_{r+i}) + e(v_{r+i+1})}{2} \right\rfloor = M - 1$. Then the eccentricity of any one of $v_{r+i}$ and $v_{r+i+1}$ is $M - 1$. Let $e(v_{r+i+1}) = M - 1$. Then $d_H(v_{r+i}, v_{r+i+1}) = M - 1$ and $e(v_{r+i}) = M$. So $v_{r+i}$ is adjacent to atleast one vertex $v_{r+j}$ in $AE_F(H)$ for some $j$, $j = i+2, i+4, i+5, \ldots, t$ whose eccentricity is $M$. Let $P_2: v_{r+i} w_1 w_2 \ldots w_{m-1} v_{r+j}$ be a shortest path between $v_{r+i}$ and $v_{r+j}$ in $H$ of length $M$. If $v_{r+j} = v_{r+i+2}$, then $v_{r+i} v_{r+i+2} \in E(AE_F(H))$. Since $v_{r+i+1} v_{r+i+2} \in E(AE_F(H))$, $v_{r+i+1} \neq w_{m-1}$ and $v_{r+i+1} \neq v_{r+i}$. So $v_{r+i+1} = w_1$ and $v_{r+i} v_{r+i+1} \in E(H)$, a contradiction to $v_{r+i} v_{r+i+1} \in E(AE_F(H))$. Assume that $i + 4 \leq j \leq t$. Since $v_{r+j-1} v_{r+j} \in E(AE_F(H))$,

$v_{r+j-1} \neq v_{r+i}$ and $v_{r+j-1} \neq w_{m-1}$ . So $v_{r+j-1} = w_1$ and $e(v_{r+j-1}) = M - 1$ . Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$ , $v_{r+i+1} \neq v_{r+j}$ and $v_{r+i+1} \neq w_1$ . So $v_{r+i+1} = w_{m-1}$ and $e(v_{r+i+1}) = M - 1$ . Since $v_{r+i+1}v_{r+i+2} \in E(AE_F(H))$ and $v_{r+i+1}v_{r+j} \in E(H)$ , $e(v_{r+i+2}) = M$ . This implies that $d_H(v_{r+i+2}, v_{r+j}) = d_H(v_{r+i+2}, v_{r+i+1})$ $+d_H(v_{r+i+1}, v_{r+j}) = \left\lfloor\frac{e(v_{r+i+2})+e(v_{r+i+1})}{2}\right\rfloor + 1 = M$ . So $v_{r+i+2}v_{r+j} \in E(AE_F(H))$ . Hence $v_{r+i}\ v_{r+i+1}v_{r+i+2}v_{r+j}v_{r+i}$ is a cycle $C_4$ in $AE_F(H)$, a contradiction. If $e(v_{r+i}) = M - 1$, then $d_H(v_{r+i}, v_{r+i+1}) = M - 1$ and $e(v_{r+i+1}) = M$. So $v_{r+i+1}$ is adjacent to atleast one vertex $v_{r+k}$ in $AE_F(H)$ for some $k$, $k = i+3, i+5, i+6, \ldots, t$ whose eccentricity is $M$. Let $P_3: v_{r+i+1}y_1y_2\ldots y_{m-1}v_{r+k}$ be a shortest path between $v_{r+i+1}$ and $v_{r+k}$ in $H$ of length $M$. If $v_{r+k} = v_{r+i+3}$, then $v_{r+i+1}v_{r+i+3} \in E(AE_F(H))$. Since $v_{r+i+2}v_{r+i+3} \in E(AE_F(H))$, $v_{r+i+2} \neq y_{m-1}$ and $v_{r+i+2} \neq v_{r+i+1}$ . So $v_{r+i+2} = y_1$ and $v_{r+i+1}v_{r+i+2} \in E(H)$ , a contradiction to $v_{r+i+1}v_{r+i+2} \in E(AE_F(H))$ . Assume that $i+5 \leq k \leq t$ . Since $v_{r+k-1}v_{r+k} \in E(AE_F(H))$ , $v_{r+k-1} \neq v_{r+i+1}$ and $v_{r+k-1} \neq y_{m-1}$ . So $v_{r+k-1} = y_1$ and $e(v_{r+j-1}) = M - 1$. Since $v_{r+i+1}v_{r+i+2} \in E(AE_F(H))$, $v_{r+i+2} \neq v_{r+k}$ and $v_{r+i+2} \neq y_1$. So $v_{r+i+2} = y_{m-1}$ and $e(v_{r+i+2}) = M - 1$ . Since $v_{r+i+2}v_{r+i+3} \in E(AE_F(H))$ and $v_{r+i+2}v_{r+k} \in E(H)$ , $e(v_{r+i+3}) = M$ . This implies that $d_H(v_{r+i+3}, v_{r+k}) = d_H(v_{r+i+3}, v_{r+i+2})$ $+d_H(v_{r+i+2}, v_{r+k})$ $= \left\lfloor\frac{e(v_{r+i+3})+e(v_{r+i+2})}{2}\right\rfloor + 1 = M$. So $v_{r+i+3}v_{r+k} \in E(AE_F(H))$ . Hence $v_{r+i+1}v_{r+i+2}v_{r+i+3}v_{r+k}v_{r+i+1}$ is a cycle $C_4$ in $AE_F(H)$ , a contradiction.

**Case 2.** $t = 3$. In this case, $v_{r+i}v_{r+i+1}v_{r+i+2}$ is a triangle in $AE_F(H)$ where $v_{r+i+3} = v_{r+i}$ for $1 \leq i \leq 3$. Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$, there is a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $\left\lfloor\frac{e(v_{r+i})+e(v_{r+i+1})}{2}\right\rfloor$.

**Case 2.1.** Suppose $e(v_{r+i}) + e(v_{r+i+1})$ is even and $\left\lfloor\frac{e(v_{r+i})+e(v_{r+i+1})}{2}\right\rfloor = M$. If $e(v_{r+i}) \neq e(v_{r+i+1})$ , then $d_H(v_{r+i}, v_{r+i+1}) < M$ . So $e(v_{r+i}) = M = e(v_{r+i+1})$ . Let $P_4: v_{r+i}w_1w_2\ldots w_{m-1}v_{r+i+1}$ be a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $M$. Since $v_{r+i+1}v_{r+i+2} \in E(AE_F(H))$, $v_{r+i+2} \neq v_{r+i}$ and $v_{r+i+2} \neq w_{m-1}$. So $v_{r+i+2} = w_1$ and $v_{r+i}v_{r+i+2} \in E(H)$, a contradiction to $v_{r+i}v_{r+i+2} \in E(AE_F(H))$.

**Case 2.2.** Suppose $e(v_{r+i}) + e(v_{r+i+1})$ is odd and $\left\lfloor\frac{e(v_{r+i})+e(v_{r+i+1})}{2}\right\rfloor = M - 1$. In this case, the eccentricity of any one of $v_{r+i}, v_{r+i+1}$ is $M - 1$ . Let $e(v_{r+i}) = M - 1$ . Then $e(v_{r+i+1}) = M$ and $e(v_{r+i+2}) = M$. Let $P_5: v_{r+i+2}w'_1w'_2\ldots w'_iw_{i+1}\ldots w_{m-1}v_{r+i+1}$ be a shortest path between $v_{r+i+2}$ and $v_{r+i+1}$ in $H$ of length $M$ . Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$, $v_{r+i} \neq v_{r+i+2}$ and $v_{r+i} \neq w_{m-1}$. So $v_{r+i} = w'_1$ and $v_{r+i+2}v_{r+i} \in E(H)$, a contradiction to $v_{r+i+2}v_{r+i} \in E(AE_F(H))$. Suppose $e(v_{r+i+1}) = M - 1$. Then $e(v_{r+i}) = M$ and hence $e(v_{r+i+2}) = M$. As in Case 2.1, $AE_F(H)$ is not equal to $G$, a contradiction. □

**Theorem 2.11.** Let $G = rK_1 \cup K_{t_1,t_2,\ldots,t_n} \cup G_1 \cup G_2 \cup \ldots \cup G_p$ be a disconnected such that $r$ and $t_i$ being postive integers, $1 \leq i \leq n$, $n \geq 2$ and each $G_j$ is a square free component and non isomorphic to $P_4$ for $1 \leq j \leq p$. Then $G$ is not a $F$-average eccentric graph.

**Proof.** Assume that $t_1 + t_2 + \ldots + t_n = t$. Let $v_1, v_2, \ldots, v_r$ be the isolated vertices of $G$, $V_i = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_{r_i}^{(i)}\}$ be the $i^{th}$ partition of $K_{t_1,t_2,\ldots,t_n}$ for $1 \leq i \leq n$ , and $v_{r+t+r_{j-1}+1}, v_{r+t+r_{j-1}+2}, \ldots,$ $v_{r+t+r_j}$ be the vertices of the component $G_j$ for $1 \leq j \leq p$ where $r_0 = 0$. Suppose there exists a graph $H$ such that $AE_F(H) = G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $v_1, v_2, \ldots, v_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$. Since $v_{t_{h_1}}^{(i)}$ and $v_{t_{h_2}}^{(j)}$ are adjacent in $AE_F(H)$ for $1 \leq h_1 \leq t_i, 1 \leq h_2 \leq t_j, i \neq j$ and $1 \leq i, j \leq n$, there is a shortest path between $v_{t_{h_1}}^{(i)}$ and $v_{t_{h_2}}^{(j)}$ in $H$ of length $\left\lceil \frac{e(v_{t_{h_1}}^{(i)})+e(v_{t_{h_2}}^{(j)})}{2} \right\rceil$.

**Case 1.** Suppose $e(v_{t_{h_1}}^{(i)}) + e(v_{t_{h_2}}^{(j)})$ is even and $\frac{e(v_{t_{h_1}}^{(i)})+e(v_{t_{h_2}}^{(j)})}{2} = M$. If $e(v_{t_{h_1}}^{(i)}) \neq e(v_{t_{h_2}}^{(j)})$, then $d_H(v_{t_{h_1}}^{(i)}, v_{t_{h_2}}^{(j)}) < M$. So $e(v_{t_{h_1}}^{(i)}) = M = e(v_{t_{h_2}}^{(j)})$. Let $P_1: v_{t_{h_1}}^{(i)} x_1 x_2 \ldots x_{m-1} v_{t_{h_2}}^{(j)}$ be a shortest path between $v_{t_{h_1}}^{(i)}$ and $v_{t_{h_2}}^{(j)}$ in $H$ of length $M$. Since $v_{t_{h_1}}^{(i)} v_{t_{h_3}}^{(k)} \in E(AE_F(H))$ for $1 \leq h_1 \leq t_i, 1 \leq h_3 \leq t_k$, $i \neq k$ and $1 \leq i, k \leq n$, $v_{t_{h_3}}^{(k)} \neq v_{t_{h_2}}^{(j)}$ and $v_{t_{h_3}}^{(k)} \neq x_1$. So $v_{t_{h_3}}^{(k)} = x_{m-1}$ and $e(v_{t_{h_3}}^{(k)}) = M - 1$. Since $v_{t_{h_2}}^{(j)} v_{t_{h_4}}^{(l)} \in E(AE_F(H))$ for $1 \leq h_2 \leq t_j, 1 \leq h_4 \leq t_l$, $j \neq l$ and $1 \leq j, l \leq n$, $v_{t_{h_4}}^{(l)} \neq v_{t_{h_1}}^{(i)}$ and $v_{t_{h_4}}^{(l)} \neq x_{m-1}$. So $v_{t_{h_4}}^{(l)} = x_1$. This implies that $d_H(v_{t_{h_4}}^{(l)}, v_{t_{h_3}}^{(k)}) = d_H(x_1, x_{m-1}) < M - 1$ and $v_{t_{h_3}}^{(k)} v_{t_{h_4}}^{(l)} \not\in E(AE_F(H))$, a contradiction.

**Case 2.** If $e(v_{t_{h_1}}^{(i)}) + e(v_{t_{h_2}}^{(j)})$ is odd and $\left\lceil \frac{e(v_{t_{h_1}}^{(i)})+e(v_{t_{h_2}}^{(j)})}{2} \right\rceil = M - 1$, then the eccentricity of any one of $v_{t_{h_1}}^{(i)}, v_{t_{h_2}}^{(j)}$ is $M - 1$. Let $e(v_{t_{h_2}}^{(j)}) = M - 1$. Then $d_H(v_{t_{h_1}}^{(i)}, v_{t_{h_2}}^{(j)}) = M - 1$ and $e(v_{t_{h_1}}^{(i)}) = M$. So $v_{t_{h_1}}^{(i)}$ is adjacent to at least one vertex $v_{t_{h_5}}^{(s)}$ in $AE_F(H)$, for some $s \neq i, 1 \leq s \leq n$ and $1 \leq h_5 \leq t_s$ whose eccentricity is $M$. As in Case 1, $AE_F(H)$ is not equal to $G$, a contradiction. Suppose $e(v_{t_{h_1}}^{(i)}) = M - 1$. Then $e(v_{t_{h_2}}^{(j)}) = M$. So $v_{t_{h_2}}^{(j)}$ is adjacent to at least one vertex $v_{t_{h_6}}^{(g)}$ in $AE_F(H)$, for some $g \neq j, 1 \leq g \leq n$ and $1 \leq h_6 \leq t_g$ whose eccentricity is $M$. As in Case 1, $AE_F(H)$ is not equal to $G$, a contradiction. Thus $G$ is not a $F$-average eccentric graph. □

**Corollary 2.12.** Let $G = rK_1 \cup K_n \cup G_1 \cup G_2 \cup \ldots \cup G_p$ be a disconnected graph such that $n \geq 3$ and $r \geq 1$, each $G_i$ is a square free component and non isomorphic to $P_4$ for $1 \leq i \leq p$ and $|V(G_i)| = r_i$ for $1 \leq i \leq p$. Then $G$ is not a $F$-average eccentric graph.
**Proof.** By taking $t_1 = t_2 = \ldots = t_n = 1$ in Theorem 2.11, $G$ is not a $F$-average eccentric graph.□

**Theorem 2.13.** If $G$ is a disconnected graph with each component complete having at least one isolated vertex, then $G$ is not a $F$-average eccentric graph.
**Proof.** Suppose $rK_1 \cup K_{r_1} \cup K_{r_2} \cup \ldots \cup K_{r_p}$ where $r_i \geq 3$, $r \geq 1$ and $1 \leq i \leq p$. Assume that $r_1 + r_2 + \ldots + r_p = t$. Let $u_1, u_2, \ldots, u_r$ be the isolated vertices of $G$, $u_{r+r_{i-1}+1}, u_{r+r_{i-1}+2}, \ldots, u_{r+r_i}$ be the vertices of $K_{r_i}$, $1 \leq i \leq p$ where $r_0 = 0$. Suppose there exists a graph $H$ such that $AE_F(H) = G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $u_1, u_2, \ldots, u_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then

by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$. If $e(u_{i'}) = d(H)$ for $1 \leq i' \leq r$, then $u_{i'}$ is not an isolated vertex in $AE_F(H)$, a contradiction. Therefore $1 < e(u_{i'}) < d(H)$ for $1 \leq i' \leq r$ and $e(u_i) = d(H)$ for some $i = r+1, r+2, \ldots, t$. Let $u_{r+r_{i-1}+j}$ be a vertex in $V(K_{r_i})$ such that $e(u_{r+r_{i-1}+j}) = d(H) = M$ and $u_{r+r_{i-1}} = u_{r+r_i}$ in $V(K_{r_i})$ for some $j = 1,2,\ldots r_i - r_{i-1}$. Since $u_{r+r_{i-1}+j}$ is adjacent to $u_{r+r_{i-1}+1}, u_{r+r_{i-1}+2}, \ldots, u_{r+r_{i-1}+j-1}, u_{r+r_{i-1}+j+1}, \ldots, u_{r+r_i}$ in $AE_F(H)$, any one in $\{u_{r+r_{i-1}+1}, u_{r+r_{i-1}+2}, \ldots, u_{r+r_{i-1}+j-1}, u_{r+r_{i-1}+j+1}, \ldots, u_{r+r_i}\}$ is the antipodal vertex of $u_{r+r_{i-1}+j}$ in $H$. Suppose $d_H(u_{r+r_{i-1}+j}, u_{r+r_{i-1}+j+1}) = M$. Let $P: u_{r+r_{i-1}+j} x_1 x_2 \ldots x_{m-1} u_{r+r_{i-1}+j+1}$ be a diametral path between $u_{r+r_{i-1}+j}$ and $u_{r+r_{i-1}+j+1}$ in $H$. Then $e(u_{r+r_{i-1}+j+1}) = M$. Since $u_{r+r_{i-1}+j} u_k \in E(AE_F(H))$ for $r + r_{i-1} + j \neq k$, $r + r_{i-1} + 1 \leq r + r_{i-1} + j, k \leq r + r_i$, $1 \leq i \leq p$, $x_h \not\in \{u_{r+r_{i-1}+1}, u_{r+r_{i-1}+2}, \ldots, u_{r+r_i}\}$ for $h = 1,2,\ldots,m-1$ and $1 \leq i \leq p$. Therefore $x_h \in \{u_1, u_2, \ldots, u_r\}$ for $h = 1,2,\ldots,m-1$ and $u_{r+r_{i-1}+j} x_{m-1}, x_1 u_{r+r_{i-1}+j+1} \in E(AE_F(H))$, a contradiction.Thus $G$ is not a $F$-average eccentric graph. □

**Theorem 2.14.** If $G$ is $rK_1 \cup C_{r_1} \cup C_{r_2} \cup \ldots \cup C_{r_p}$, $r_i \geq 3$, $r \geq 1$ and $1 \leq i \leq p$, then $G$ is not a $F$-average eccentric graph.

**Proof.** Assume that $r_1 + r_2 + \ldots + r_p = t$. Let $u_1, u_2, \ldots, u_r$ be the isolated vertices of $G$, $u_{r+r_{i-1}+1}, u_{r+r_{i-1}+2}, \ldots, u_{r+r_i}$ be the vertices on the cycle $C_{r_i}$, $1 \leq i \leq p$ where $r_0 = 0$. If $G$ has a triangle of length 3, then by Theorem 2.13, $G$ is not a $F$-average eccentric graph. Let $t \geq 4$. Suppose there exists a graph $H$ such that $AE_F(H) = G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $u_1, u_2, \ldots, u_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$. If $e(u_{i'}) = d(H)$ for $1 \leq i' \leq r$, then $u_{i'}$ is not an isolated vertex in $AE_F(H)$, a contradiction. Therefore $1 < e(u_{i'}) < d(H)$ for $1 \leq i' \leq r$ and $e(u_i) = d(H)$ for some $i = r+1, r+2, \ldots, t$. Let $u_{r+r_{i-1}+j}$ be a vertex in $V(C_{r_i})$ such that $e(u_{r+r_{i-1}+j}) = d(H) = M$ and $u_{r+r_{i-1}} = u_{r+r_i}$ in $V(C_{r_i})$ for some $j = 1,2,\ldots r_i - r_{i-1}$. Since $u_{r+r_{i-1}+j}$ is adjacent to $u_{r+r_{i-1}+j-1}$ and $u_{r+r_{i-1}+j+1}$ only in $AE_F(H)$, any one in $\{u_{r+r_{i-1}+j-1}, u_{r+r_{i-1}+j+1}\}$ is the antipodal vertex of $u_{r+r_{i-1}+j}$ in $H$. Suppose $d_H(u_{r+r_{i-1}+j}, u_{r+r_{i-1}+j+1}) = M$. Let $P: u_{r+r_{i-1}+j} x_1 x_2 \ldots x_{m-1}$
$u_{r+r_{i-1}+j+1}$ be a diametral path between $u_{r+r_{i-1}+j}$ and $u_{r+r_{i-1}+j+1}$ in $H$. Then $e(u_{r+r_{i-1}+j+1}) = M$. Since $u_{r+r_{i-1}+j-1} u_{r+r_{i-1}+j} \in E(AE_F(H))$, $u_{r+r_{i-1}+j-1} \neq x_1$ and $u_{r+r_{i-1}+j-1} \neq u_{r+r_{i-1}+j+1}$. So $u_{r+r_{i-1}+j-1} = x_{m-1}$ and $e(u_{r+r_{i-1}+j-1}) = M - 1$. Since $u_{r+r_{i-1}+j+1} u_{r+r_{i-1}+j+2} \in E(AE_F(H))$, $u_{r+r_{i-1}+j+2} \neq x_{m-1}$ and $u_{r+r_{i-1}+j+2} \neq u_{r+r_{i-1}+j}$. So $u_{r+r_{i-1}+j+2} = x_1$ and $e(u_{r+r_{i-1}+j+2}) = M - 1$.

**Case 1.** Suppose $G$ has a cycle $C_{r_i}$ of length $\geq 5$. Since $e(u_{r+r_{i-1}+j+2}) = M - 1$, $u_{r+r_{i-1}+j+3} \in V(H) - \{u_{r+r_{i-1}+j}, x_1, x_2, \ldots, x_{m-1}, u_{r+r_{i-1}+j+1}\}$ .If $u_{r+r_{i-1}+j+3} \in \{u_1, u_2, \ldots, u_r\}$, then any one of $u_1, u_2, \ldots, u_r$ is not an isolated vertex in $AE_F(H)$, a contradiction. So $u_{r+r_{i-1}+j+3} \in \{u_{r+1}, u_{r+2}, \ldots, u_t\} - \{u_{r+r_{i-1}+j}, x_1, x_2, \ldots, x_{m-1}, u_{r+r_{i-1}+j+1}$. Then $d_H(u_{r+r_{i-1}+j}, u_{r+r_{i-1}+j+3}) = M$. Hence $u_{r+r_{i-1}+j} u_{r+r_{i-1}+j+3} \in E(AE_F(H))$, a contradiction.

**Case 2.** Suppose $G$ has a cycle $C_{r_i}$ of length 4. Then $u_{r+r_{i-1}+j-1}u_{r+r_{i-1}+j}u_{r+r_{i-1}+j+1}$ $u_{r+r_{i-1}+j+2}u_{r+r_{i-1}+j-1}$ is a cycle $C_{r_i} = C_4$ in $AE_F(H)$ and $u_{r+r_{i-1}} = u_{r+r_i}$ in $V(C_{r_i})$ for $j = 1,2,3,4$. So $d_H(u_{r+r_{i-1}+j+2}, u_{r+r_{i-1}+j-1}) = d_H(x_1, x_{m-1}) < M-1$ and $u_{r+r_{i-1}+j-1}u_{r+r_{i-1}+j+2} \not\in E(AE_F(H))$, a contradiction. □

**Theorem 2.15.** Let $G = rK_1 \cup C_t \cup G_1 \cup G_2 \cup \ldots \cup G_p$ be a disconnected graph such that $r \geq 1$, $t \geq 3$ and each $G_i$ is a square free component and non isomorphic to $P_4$ for $1 \leq i \leq p$. Then $G$ is not a $F$-average eccentric graph.

**Proof.** Let $v_1, v_2, \ldots, v_r$ be the isolated vertices, $v_{r+1}, v_{r+2}, \ldots, v_{r+t}$ be the vertices on the cycle $C_t$ and $v_{r+t+r_{i-1}+1}, v_{r+t+r_{i-1}+2}, \ldots, v_{r+t+r_i}$ be the vertices of the component $G_i$, $1 \leq i \leq p$ where $r_0 = 0$. Suppose there exists a graph $H$ such that $AE_F(H) = G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $v_1, v_2, \ldots, v_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$.

**Case 1.** $t = 3$ or $t \geq 5$. Then by Theorem 2.10, $G$ is not a $F$-average eccentric graph.

**Case 2.** $t = 4$. In this case, $v_{r+i}v_{r+i+1}v_{r+i+2}v_{r+i+3}v_{r+i}$ is a cycle $C_4$ in $AE_F(H)$ where $v_{r+i+4} = v_{r+i}$ for $1 \leq i \leq 4$. Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$, there is a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $\left\lfloor \frac{e(v_{r+i})+e(v_{r+i+1})}{2} \right\rfloor$.

**Case 2.1.** Suppose $e(v_{r+i}) + e(v_{r+i+1})$ is even and $\left\lfloor \frac{e(v_{r+i})+e(v_{r+i+1})}{2} \right\rfloor = M$.
If $e(v_{r+i}) \neq e(v_{r+i+1})$, then $d_H(v_{r+i}, v_{r+i+1}) < M$. So $e(v_{r+i}) = M = e(v_{r+i+1})$. Let $P_1\colon v_{r+i}w_1w_2\ldots w_{m-1}v_{r+i+1}$ be a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $M$. Since $v_{r+i+1}v_{r+i+2} \in E(AE_F(H))$, $v_{r+i+2} \neq v_{r+i}$ and $v_{r+i+2} \neq w_{m-1}$. So $v_{r+i+2} = w_1$ and $e(v_{r+i+2}) = M-1$. Since $v_{r+i}v_{r+i+3} \in E(AE_F(H))$, $v_{r+i+3} \neq v_{r+i+1}$ and $v_{r+i+3} \neq w_1$. So $v_{r+i+3} = w_{m-1}$ and $e(v_{r+i+3}) = M-1$. This implies that $d_H(v_{r+i+2}, v_{r+i+3}) = d_H(w_1, w_{m-1}) < M-1$ and $v_{r+i+2}v_{r+i+3} \not\in E(AE_F(H))$, a contradiction.

**Case 2.2.** Suppose $e(v_{r+i}) + e(v_{r+i+1})$ is odd and $\left\lfloor \frac{e(v_{r+i})+e(v_{r+i+1})}{2} \right\rfloor = M-1$. In this case, the eccentricity of any one of $v_{r+i}, v_{r+i+1}$ is $M-1$. Let $e(v_{r+i}) = M-1$. Then $e(v_{r+i+1}) = M$. Since $v_{r+i+1}$ is adjacent to $v_{r+i}$ and $v_{r+i+2}$ only, $e(v_{r+i+2}) = M$. Let $P_2\colon v_{r+i+2}w'_1w'_2\ldots w'_iw_{i+1}\ldots w_{m-1}v_{r+i+1}$ be a shortest path between $v_{r+i+2}$ and $v_{r+i+1}$ in $H$ of length $M$. Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$, $v_{r+i} \neq v_{r+i+2}$ and $v_{r+i} \neq w_{m-1}$. So $v_{r+i} = w'_1$ and $e(v_{r+i}) = M-1$. Since $v_{r+i+2}v_{r+i+3} \in E(AE_F(H))$, $v_{r+i+3} \neq v_{r+i+1}$ and $v_{r+i+3} \neq w'_1$. So $v_{r+i+3} = w_{m-1}$ and $e(v_{r+i+3}) = M-1$. This implies that $d_H(v_{r+i}, v_{r+i+3}) = d_H(w'_1, w_{m-1}) < M-1$ and $v_{r+i}v_{r+i+3} \not\in E(AE_F(H))$, a contradiction.
Suppose $e(v_{r+i+1}) = M-1$. Then $e(v_{r+i}) = M$ and hence $e(v_{r+i+3}) = M$. As in Case 2.1, $AE_F(H)$ is not equal to $G$, a contradiction. □

**Theorem 2.16.** Let $G = rK_1 \cup T_t \cup G_1 \cup G_2 \cup \ldots \cup G_p$ be a disconnected such that $r \geq 1$, a tree $T_t$ on $t \geq 5$ vertices as a component having a path on length 4 and each $G_i$ is a square free component and non isomorphic to $P_4$ for $1 \leq i \leq p$. Then $G$ is not a $F$-average eccentric graph.

**Proof.** Let $v_1, v_2, \ldots, v_r$ be the isolated vertices, $v_{r+1}, v_{r+2}, \ldots, v_{r+t}$ be the vertices on the cycle $T_t$ and $v_{r+t+r_{i-1}+1}, v_{r+t+r_{i-1}+2}, \ldots, v_{r+t+r_i}$ be the vertices of the component $G_i$, $1 \leq i \leq p$ where $r_0 = 0$. Suppose there exists a graph $H$ such that $AE_F(H) = G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G} \in F_{12}$. So $H$ is connected. By the definition, each of $v_1, v_2, \ldots, v_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H) \geq 2$. Let $v_{r+i}, v_{r+j}, v_{r+k}, v_{r+l}, v_{r+m}$ be the consecutive adjacent vertices of $T_t$. Since $v_{r+j}$ and $v_{r+k}$ are adjacent in $AE_F(H)$, there is a shortest path between $v_{r+j}$ and $v_{r+k}$ in $H$ of length $\left\lfloor \frac{e(v_{r+j})+e(v_{r+k})}{2} \right\rfloor$.

**Case 1.** Suppose $e(v_{r+j}) + e(v_{r+k})$ is even or odd.

**Case 1.1.** Suppose $e(v_{r+j}) + e(v_{r+k})$ is even and $\frac{e(v_{r+j})+e(v_{r+k})}{2} = M$. If $e(v_{r+j}) \neq e(v_{r+k})$, then $d_H(v_{r+j}, v_{r+k}) < M$. So $e(v_{r+j}) = M = e(v_{r+k})$. Let $P_1: v_{r+j}x_1x_2\ldots x_{m-1}v_{r+k}$ be a shortest path between $v_{r+j}$ and $v_{r+k}$ in $H$ of length $M$. Since $v_{r+k}v_{r+l} \in E(AE_F(H))$, $v_{r+l} \neq v_{r+j}$ and $v_{r+l} \neq x_{m-1}$. So $v_{r+l} = x_1$ and $e(v_{r+l}) = M - 1$. Suppose $e(v_{r+m}) = M - 1$. Let $P_2: v_{r+l}x_2x_3\ldots x_i x'_{i+1}\ldots x'_{m-1}v_{r+m}$ be a shortest path between $v_{r+l}$ and $v_{r+m}$ in $H$ of length $M - 1$. This implies that $d_H(v_{r+j}, v_{r+m}) = d_H(v_{r+j}, v_{r+l}) + d_H(v_{r+l}, v_{r+m}) = M$ and $v_{r+j}v_{r+m} \in E(AE_F(H))$, a contradiction. If $e(v_{r+m}) = M$, then $d_H(v_{r+j}, v_{r+m}) > M$ which is impossible since $e(v_{r+j}) = M$.

**Case 1.2.** If $e(v_{r+j}) + e(v_{r+k})$ is odd and $\left\lfloor \frac{e(v_{r+j})+e(v_{r+k})}{2} \right\rfloor = M - 1$, then the eccentricity of any one of $v_{r+j}, v_{r+k}$ is $M - 1$. Let $e(v_{r+k}) = M - 1$. Then $d_H(v_{r+j}, v_{r+k}) = M - 1$ and $e(v_{r+j}) = M$. So $v_{r+j}$ is adjacent to at least one vertex $v_{r+j^*} \in T_t$ whose eccentricity is $M$. Let $P_2: v_{r+j}y_1y_2\ldots y_{m-1}v_{r+j^*}$ be a shortest path between $v_{r+j}$ and $v_{r+j^*}$ in $H$ of length $M$. Since $v_{r+j}v_{r+k} \in E(AE_F(H))$, $v_{r+k} \neq v_{r+j^*}$ and $v_{r+k} \neq y_1$. So $v_{r+k} = y_{m-1}$. Since $v_{r+k}v_{r+l} \in E(AE_F(H))$, $v_{r+l} \neq v_{r+j^*}$ and $v_{r+l} \neq v_{r+j}$. Since $e(v_{r+k}) = M - 1$, $d_H(v_{r+k}, v_{r+l}) = M - 1$ and $e(v_{r+l}) = M$. This implies that $d_H(v_{r+l}, v_{r+j^*}) = d_H(v_{r+l}, v_{r+k}) + d_H(v_{r+k}, v_{r+j^*}) = M$ and $v_{r+l}v_{r+j^*} \in E(AE_F(H))$. Hence $v_{r+j}v_{r+k}v_{r+l}v_{r+j^*}v_{r+j}$ is a cycle $C_4$ in $AE_F(H)$, a contradiction. Suppose $e(v_{r+j}) = M - 1$. Then $e(v_{r+k}) = M$. So $v_{r+k}$ is adjacent to at least one vertex $v_{r+k^*} \in T_t$ whose eccentricity is $M$. Let $P_3: v_{r+k}y'_1y'_2\ldots y'_{m-1}v_{r+k^*}$ be a shortest path between $v_{r+k}$ and $v_{r+k^*}$ in $H$ of length $M$. Since $v_{r+k}v_{r+j} \in E(AE_F(H))$, $v_{r+j} \neq v_{r+k^*}$ and $v_{r+j} \neq y'_1$. So $v_{r+j} = y'_{m-1}$. Since $e(v_{r+j}) = M - 1$, $d_H(v_{r+j}, v_{r+i}) = M - 1$ and $e(v_{r+i}) = M$. This implies that $d_H(v_{r+i}, v_{r+k^*}) = d_H(v_{r+i}, v_{r+j}) + d_H(v_{r+j}, v_{r+k^*}) = M$ and $v_{r+i}v_{r+k^*} \in E(AE_F(H))$. Hence $v_{r+k}v_{r+k^*}v_{r+i}v_{r+j}v_{r+k}$ is a cycle $C_4$ in $AE_F(H)$, a contradiction.

**Case 2.** If either $e(v_{r+k}) + e(v_{r+l})$ is even or odd, then as in Case 1, $AE_F(H)$ is not equal to $G$.

**Case 3.** Suppose $e(v_{r+i}) + e(v_{r+j})$ is even or odd. If $v_{r+i}$ is not a pendant vertex, then as in Case 1, $AE_F(H)$ is not equal to $G$. Suppose $v_{r+i}$ is a pendant vertex.

**Case 3.1.** If $e(v_{r+i}) + e(v_{r+j})$ is even and $\left\lfloor \frac{e(v_{r+i})+e(v_{r+j})}{2} \right\rfloor = M$. If $e(v_{r+i}) \neq e(v_{r+j})$, then $d_H(v_{r+i}, v_{r+j}) < M$. So $e(v_{r+i}) = M = e(v_{r+j})$. Let $P_4: v_{r+i}z_1z_2\ldots z_{m-1}v_{r+j}$ be a shortest path between $v_{r+i}$ and $v_{r+j}$ in $H$ of length $M$. Then $d_H(v_{r+i}, z_{m-1}) = M - 1$ and $e(z_{m-1}) = M - 1$. Hence $v_{r+i}z_{m-1} \in E(AE_F(H))$, a contradiction to the fact $v_{r+i}$ is a pendant vertex.

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**Case 3.2.** If $e(v_{r+i})+e(v_{r+j})$ is odd and $\left\lfloor\frac{e(v_{r+i})+e(v_{r+j})}{2}\right\rfloor=M-1$, then the eccentricity of any one of $v_{r+i},v_{r+j}$ is $M-1$. Let $e(v_{r+i})=M-1$. Then $d_H(v_{r+i},v_{r+j})=M-1$ and $e(v_{r+j})=M$. So $v_{r+j}$ is adjacent to at least one vertex $v_{r+j^*}\in T_t$ whose eccentricity is $M$. Since $P_2: v_{r+j}y_1y_2\ldots y_{m-1}v_{r+j^*}$ is a shortest path between $v_{r+j}$ and $v_{r+j^*}$ in $H$ of length $M$ and $v_{r+j}v_{r+k}\in E(AE_F(H))$, $v_{r+k}\neq v_{r+j^*}$ and $v_{r+k}\neq y_1$. So $v_{r+k}=y_{m-1}$. Since $v_{r+k}v_{r+l}\in E(AE_F(H))$, $v_{r+l}\neq v_{r+j^*}$ and $v_{r+l}\neq v_{r+j}$. Since $e(v_{r+k})=M-1$, $d_H(v_{r+k},v_{r+l})=M-1$ and $e(v_{r+l})=M$. This implies that $d_H(v_{r+l},v_{r+j^*})=d_H(v_{r+l},v_{r+k})+d_H(v_{r+k},v_{r+j^*})=M$ and $v_{r+l}v_{r+j^*}\in E(AE_F(H))$. Hence $v_{r+j}v_{r+k}v_{r+l}v_{r+j^*}v_{r+j}$ is a cycle $C_4$ in $AE_F(H)$, a contradiction. Suppose $e(v_{r+i})=M$. Since $v_{r+i}$ is a pendant vertex, $e(v_{r+j})=M$, a contradiction to $e(v_{r+j})=M-1$.

**Case 4.** Suppose $e(v_{r+l})+e(v_{r+m})$ is even or odd. If $v_{r+m}$ is not a pendant vertex, then as in Case 1, $AE_F(H)$, is not equal to $G$. Suppose $v_{r+m}$ is a pendant vertex. Then as in Case 3.1 and 3.2, $AE_F(H)$ is not equal to $G$. Thus $G$ is not a $F$-average eccentric graph. □

**Proposition 2.17.** Let $G=rK_1\cup L_t\cup G_1\cup G_2\cup\ldots\cup G_p$ be a disconnected such that $r\geq 1$, a ladder $L_t$ as a component with $t\geq 2$ steps, each $G_i$ is a square free component and non isomorphic to $P_4$ for $1\leq i\leq p$ and $|V(G_i)|=r_i$ for $i=1,2,\ldots,p$. Then $G$ is not a $F$-average eccentric graph.

**Proof.** Let $v_1,v_2,\ldots,v_r$ be the isolated vertices of $G$, $v_{r+1},v_{r+2},\ldots,v_{r+t}$, $w_{r+1},\ldots,w_{r+t}$ be the vertices of the ladder $L_t$ and $v_{r+t+r_{i-1}+1},v_{r+t+r_{i-1}+2},\ldots,v_{r+t+r_i}$ be the vertices of the component $G_i$, $1\leq i\leq p$ where $r_0=0$. Suppose there exists a graph $H$ such that $AE_F(H)=G$. If $H$ is disconnected, then each component of $\overline{AE_F(H)}$ is complete, a contradiction to $\overline{G}\in F_{12}$. So $H$ is connected. By the definition, each of $v_1,v_2,\ldots,v_r$ has no $F$- average eccentric vertices in $H$. If $H$ has a full degree vertex, then by Theorem A, $AE_F(H)$ has a full degree vertex, a contradiction. So $r(H)\geq 2$. Since $v_{r+i}$ and $v_{r+i+1}$ are adjacent in $AE_F(H)$, there is a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $\left\lfloor\frac{e(v_{r+i})+e(v_{r+i+1})}{2}\right\rfloor$. If $t=2$, then $L_t=C_4$ and by Theorem A, the result follows. So $t\geq 3$.

**Case 1.** $e(v_{r+i})+e(v_{r+i+1})$ is even or odd.

**Case 1.1.** $e(v_{r+i})+e(v_{r+i+1})$ is even and $\frac{e(v_{r+i})+e(v_{r+i+1})}{2}=M$. If $e(v_{r+i})\neq e(v_{r+i+1})$, then $d_H(v_{r+i},v_{r+i+1})<M$. So $e(v_{r+i})=M=e(v_{r+i+1})$. Let $P_1: v_{r+i}x_1x_2\ldots x_{m-1}v_{r+i+1}$ be a shortest path between $v_{r+i}$ and $v_{r+i+1}$ in $H$ of length $M$. Since $v_{r+i+1}w_{r+i+1}\in E(AE_F(H))$, $w_{r+i+1}\neq v_{r+i}$ and $w_{r+i+1}\neq x_{m-1}$. So $w_{r+i+1}=x_1$ and $e(w_{r+i+1})=M-1$. Since $v_{r+i}w_{r+i}\in E(AE_F(H))$, $w_{r+i}\neq v_{r+i+1}$ and $w_{r+i}\neq x_1$. So $w_{r+i}=x_{m-1}$ and $e(w_{r+i})=M-1$. This implies that $d_H(w_{r+i},w_{r+i+1})=d_H(x_{m-1},x_1)<M-1$ and $w_{r+i}w_{r+i+1}\not\in E(AE_F(H))$, a contradiction.

**Case 1.2.** If $e(v_{r+i})+e(v_{r+i+1})$ is odd and $\left\lfloor\frac{e(v_{r+i})+e(v_{r+i+1})}{2}\right\rfloor=M-1$, then the eccentricity of any one of $v_{r+i},v_{r+i+1}$ is $M-1$. Let $e(v_{r+i+1})=M-1$. Then $d_H(v_{r+i},v_{r+i+1})=M-1$ and $e(v_{r+i})=M$. Since $v_{r+i}$ is adjacent to $v_{r+i-1},w_{r+i}$ and $v_{r+i+1}$ only, the eccentricity of any one of $v_{r+i-1},w_{r+i}$ is $M$. Suppose $e(v_{r+i-1})=M$. Let $P_2: v_{r+i}x_1x_2\ldots x_i x'_{i+1}\ldots x'_{m-1}v_{r+i-1}$ be a shortest path between $v_{r+i}$ and $v_{r+i-1}$ in $H$ of length $M$. Since $v_{r+i}w_{r+i}\in E(AE_F(H))$, $w_{r+i}\neq v_{r+i-1}$ and $w_{r+i}\neq x_1$. So $w_{r+i}=x'_{m-1}$ and $e(w_{r+i})=M-1$. Since $v_{r+i-1}w_{r+i-1}\in E(AE_F(H))$, $w_{r+i-1}\neq v_{r+i}$ and $w_{r+i-1}\neq$

$x'_{m-1}$. So $w_{r+i-1} = x_1$ and $e(w_{r+i-1}) = M-1$. This implies that $d_H(w_{r+i-1}, w_{r+i}) = d_H(x_1, x'_{m-1}) < M-1$ and $w_{r+i-1}w_{r+i} \not\subseteq E(AE_F(H))$, a contradiction. Suppose $e(w_{r+i}) = M$. Let $P_3: v_{r+i}x_1x_2 \ldots x_j x''_{j+1} \ldots x''_{m-1}\ w_{r+i}$ be a shortest path between $v_{r+i}$ and $w_{r+i}$ in $H$ of length $M$. Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$, $v_{r+i+1} \neq w_{r+i}$ and $v_{r+i+1} \neq x_1$. So $v_{r+i+1} = x''_{m-1}$ and $e(v_{r+i+1}) = M-1$. Since $w_{r+i}w_{r+i+1} \in E(AE_F(H))$, $w_{r+i+1} \neq v_{r+i}$ and $w_{r+i+1} \neq x''_{m-1}$. So $w_{r+i+1} = x_1$ and $e(w_{r+i+1}) = M-1$. This implies that $d_H(w_{r+i+1}, v_{r+i+1}) = d_H(x_1, x''_{m-1}) < M-1$ and $w_{r+i+1}v_{r+i+1} \not\subseteq E(AE_F(H))$ a contradiction. If $i=1$, then $v_{r+1}$ is adjacent to $w_{r+1}$ and $v_{r+2}$ only and $e(w_{r+1}) = M$. Since $v_{r+1}v_{r+2} \in E(AE_F(H))$, by the path $P_3$, $v_{r+2} \neq w_{r+1}$ and $v_{r+2} \neq x_1$. So $v_{r+2} = x''_{m-1}$ and $e(v_{r+2}) = M-1$. Since $w_{r+1}w_{r+2} \in E(AE_F(H))$, $w_{r+2} \neq v_{r+1}$ and $w_{r+2} \neq x''_{m-1}$. So $w_{r+2} = x_1$ and $e(w_{r+2}) = M-1$. This implies that $d_H(w_{r+2}, v_{r+2}) = d_H(x_1, x''_{m-1}) < M-1$ and $w_{r+2}v_{r+2} \not\subseteq E(AE_F(H))$, a contradiction.

**Case 2.** If either $e(w_{r+i}) + e(w_{r+i+1})$ is even or odd, then as in Case 1, $AE_F(H) \neq G$.

**Case 3.** Suppose $e(v_{r+i}) + e(w_{r+i})$ is even or odd.

**Case 3.1.** If $e(v_{r+i}) + e(w_{r+i})$ is even and $\frac{e(v_{r+i})+e(w_{r+i})}{2} = M$. If $e(v_{r+i}) \neq e(w_{r+i})$, then $d_H(v_{r+i}, w_{r+i}) < M$. So $e(v_{r+i}) = M = e(w_{r+i})$. Let $P_4: v_{r+i}y_1y_2 \ldots y_{m-1}w_{r+i}$ be a shortest path between $v_{r+i}$ and $w_{r+i}$ in $H$ of length $M$. Since $v_{r+i}v_{r+i+1} \in E(AE_F(H))$, $v_{r+i+1} \neq w_{r+i}$ and $v_{r+i+1} \neq y_1$. So $v_{r+i+1} = y_{m-1}$ and $e(v_{r+i+1}) = M-1$. Since $w_{r+i}w_{r+i+1} \in E(AE_F(H))$, $w_{r+i+1} \neq v_{r+i}$ and $w_{r+i+1} \neq y_{m-1}$. So $w_{r+i+1} = y_1$ and $e(w_{r+i+1}) = M-1$. This implies that $d_H(v_{r+i+1}, w_{r+i+1}) = d_H(y_1, y_{m-1}) < M-1$ and $v_{r+i+1}w_{r+i+1} \not\subseteq E(AE_F(H))$, a contradiction. If $i=t$, then $v_{r+t}$ is adjacent to $v_{r+t-1}$ and $w_{r+t}$ only and $e(w_{r+t}) = M$. Since $v_{r+t-1}v_{r+t} \in E(AE_F(H))$, by the path $P_4$, $v_{r+t-1} \neq y_1$ and $v_{r+t-1} \neq w_{r+i}$. So $v_{r+t-1} = y_{m-1}$ and $e(v_{r+t-1}) = M-1$. Since $w_{r+t-1}w_{r+t} \in E(AE_F(H))$, $w_{r+t-1} \neq y_{m-1}$ and $w_{r+t-1} \neq v_{r+t}$. So $w_{r+t-1} = y_1$ and $e(w_{r+t-1}) = M-1$. This implies that $d_H(w_{r+t-1}, v_{r+t-1}) = d_H(y_1, y_{m-1}) < M-1$ and $v_{r+t-1}w_{r+t-1} \not\subseteq E(AE_F(H))$,a contradiction

**Case 3.2.** If $e(v_{r+i}) + e(w_{r+i})$ is odd and $\left\lfloor\frac{e(v_{r+i})+e(w_{r+i})}{2}\right\rfloor = M-1$, then the eccentricity of any one of $v_{r+i}, w_{r+i}$ is $M-1$. Let $e(w_{r+i}) = M-1$. Then $d_H(v_{r+i}, w_{r+i}) = M-1$ and $e(v_{r+i}) = M$. Since $v_{r+i}$ is adjacent to $v_{r+i-1}, w_{r+i}$ and $v_{r+i+1}$ only, the eccentricity of any one of $v_{r+i-1}, v_{r+i+1}$ is $M$ .Suppose $e(v_{r+i-1}) = M$ . Let $P_5: v_{r+i}y_1y_2 \ldots y_j y'_{j+1} \ldots y'_{m-1}v_{r+i-1}$ be a shortest path between $v_{r+i}$ and $v_{r+i-1}$ in $H$ of length $M$. Since $v_{r+i-1}w_{r+i-1} \in E(AE_F(H))$, $w_{r+i-1} \neq v_{r+i}$ and $w_{r+i-1} \neq y'_{m-1}$. So $w_{r+i-1} = y_1$ and $e(w_{r+i-1}) = M-1$. Since $v_{r+i}w_{r+i} \in E(AE_F(H))$, $w_{r+i} \neq v_{r+i-1}$ and $w_{r+i} \neq y_1$. So $w_{r+i} = y'_{m-1}$ and $e(w_{r+i}) = M-1$. This implies that $d_H(w_{r+i-1}, w_{r+i}) = d_H(y_1, y'_{m-1}) < M-1$ and $w_{r+i-1}w_{r+i} \not\subseteq E(AE_F(H))$ , a contradiction. Suppose $e(v_{r+i+1}) = M$. Then by case 1.1, $w_{r+i-1}w_{r+i} \not\subseteq E(AE_F(H))$, a contradiction. Suppose $e(v_{r+i}) = M-1$. Then $e(w_{r+i}) = M$. Since $w_{r+i}$ is adjacent to $w_{r+i-1}, v_{r+i}$ and $w_{r+i+1}$ only, the eccentricity of any one of $w_{r+i-1}, w_{r+i+1}$ is $M$. Suppose $e(w_{r+i-1}) = M$. Let $P_6: w_{r+i}y''_1y''_2 \ldots y''_{m-1}w_{r+i-1}$ be a shortest path between $w_{r+i}$ and $w_{r+i-1}$ in $H$ of length $M$. Since $w_{r+i-1}v_{r+i-1} \in E(AE_F(H))$, $v_{r+i-1} \neq w_{r+i}$ and $v_{r+i-1} \neq y''_{m-1}$. So $v_{r+i-1} = y''_1$ and $e(v_{r+i-1}) = M-1$. Since $w_{r+i}v_{r+i} \in E(AE_F(H))$, $v_{r+i} \neq w_{r+i-1}$ and $v_{r+i} \neq y''_1$. So $v_{r+i} = y''_{m-1}$ and $e(v_{r+i}) = M-1$. This implies that $d_H(v_{r+i-1}, v_{r+i}) = d_H(y''_1, y''_{m-1}) < M-1$ and hence $v_{r+i-1}v_{r+i} \not\subseteq E(AE_F(H))$ a contradiction. Suppose $e(w_{r+i+1}) = M$ . By case 1.1, $v_{r+i}v_{r+i+1} \not\subseteq E(AE_F(H))$ , a contradiction. □

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