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Design Of Robust LQR Control For Nonlinear System Using Adaptive Dynamic Programming

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Abstract:* One of the important challenges in the design of LQR for real time applications is the optimal choice state and input weighting matrices (Q and R), which play a vital role in determining the performance and optimality of the controller. Commonly, trial and error approach is employed for selecting the weighting matrices, which not only burdens the design but also results in non-optimal response. Hence, to choose the elements of Q and R matrices optimally, Adaptive Dynamic Programming (ADP) algorithm is used for selecting the most suitable Q and R matrices by iteration which reduces the performance index of the system to be considered. However, stability is only a bare minimum requirement in a system design. Ensuring optimality guarantees the stability of the nonlinear system. Dynamic programming is a very useful tool in solving optimization and optimal control problems by employing the principle of optimality. There are several spectrums about the dynamic programming. One can consider discrete-time systems or continuous-time systems, linear systems or nonlinear systems, time-invariant systems or time-varying systems, deterministic systems or stochastic systems. The inverted pendulum is a standard benchmark control problem and for the control of which numerous control algorithms have evolved over the ages. The main objective of this project is to design a robust linear quadratic regulator (RLQR) for nonlinear system using adaptive dynamic programming and to propose an optimal tracking control approach based on adaptive dynamic programming (ADP) algorithm in order to solve the linear quadratic regulation problems for nonlinear systems in an online fashion.

Keywords—LQR, weighting matrices, Adaptive Dynamic Programming, nonlinear system, principal of optimality, online fashion

1. INTRODUCTION

In real time all the systems are affected by various uncertainties due to modeling error, external disturbance and parameter variations. Controlling such a dynamical system is difficult and there arise the need for robust controllers. These controllers will achieve the desired performance of system despite uncertainties.

The inverted pendulum is a standard benchmark control problem and for the control of which numerous control algorithms have evolved over the ages. Linear quadratic regulator (LQR) is one among the control algorithm. One of the challenging problems in the design of LQR is the choice of Q and R matrices. Conventionally, the weights of a LQR controller are chosen based on a trial and error approach to determine the optimum state feedback controller gains. However, it is often time consuming and tedious to tune the controller gains via a trial and error method. To avoid these problems, an iterative approach for the selection of Q and R matrices has been introduced in this paper. This iterative approach leads to adaptive dynamic programming (ADP) where the iteration is done to minimize the performance index or cost function by selecting the optimal Q and R matrices for the given system.

Derong Liu and Qinglai Wei [1] developed a new Policy Iteration Adaptive Dynamic Programming Algorithm for Discrete-Time Nonlinear Systems for solving the infinite horizon optimal control problem of nonlinear systems. Their idea is to use an iterative ADP technique to obtain the iterative control law, which optimizes the iterative performance index function.

Qingqing Xie, B. Luo, and F. Tan [2] presented a new Discrete-Time LQR Optimal Tracking Control Problems Using Approximate Dynamic Programming Algorithm with Disturbance. The iterative ADP algorithm via Heuristic Dynamic Programming (HDP) technique is introduced to solve the value function of the controlled system. To verify its robustness, disturbance is added to the controlled system.

Yang Liu, Yanhong Luo and Huaguang Zhang [3] proposed an Adaptive Dynamic Programming for Discrete-time LQR Optimal Tracking Control Problems with Unknown Dynamics. An optimal tracking control approach based on adaptive dynamic programming (ADP) algorithm is proposed to solve the linear quadratic regulation (LQR) problems for unknown discrete-time systems in an online fashion. It is shown that the proposed ADP algorithm solves the LQR without requiring any knowledge of the system dynamics. The simulation results show the convergence and effectiveness of the proposed control scheme.

Vinodh Kumar E and Jovitha Jerome [4] developed a Robust LQR Controller Design for Stabilizing and Trajectory Tracking of Inverted Pendulum. It describes the method for stabilizing and trajectory tracking of Self Erecting Single Inverted Pendulum (SESIP) using Linear Quadratic Regulator (LQR). A robust LQR is proposed in this paper not only to stabilize the pendulum in upright position but also to make the cart system to track the given reference signal even in the presence of disturbance. An optimal LQR controller with well-tuned weighting matrices is implemented to stabilize the pendulum in the vertical position. As a future work, to further reduce the oscillation amplitude and frequency, friction compensation schemes can be incorporated in the controller strategy.

Chaiporn Wongsathan and Chanapoom Sirima [5] developed an Application of GA to Design LQR Controller for an Inverted Pendulum System. Genetic Algorithm (GA) is applied to design weighting matrices in Linear Quadratic Regulator (LQR) for an Inverted Pendulum System (IPS). Feedback gain settings of the system are obtained by minimizing the performance index using GA to optimize the weight matrices of LQR. In the future, the design idea can be extended to the control method such as fuzzy logic controller, neuro-fuzzy controller for the non-linear model of IPS which needs to be studied ulterior.

System description and modeling

A. *Inverted Pendulum Sysyem*

The inverted pendulum is a nonlinear, unstable, under actuated system. The system is under actuated as it has two degrees of motion with a single input and such systems are difficult to control. The output is the linear motion of the cart and the angular motion of the pendulum. Because of this nature of the system, they are selected for studying various modern control problems. The schematic representation of the inverted pendulum on a moving cart system is shown in Fig. 1.

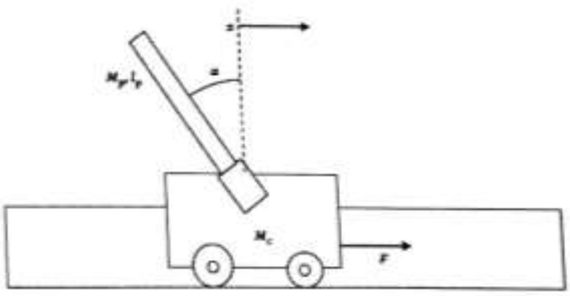

Fig. 1. Cart-Inverted Pendulum System

The cart-inverted pendulum system consists of a pendulum of mass and length attached to the cart of mass and the cart in turn is attached to a motor that drives the cart along the horizontal track by means of gear arrangement. The mass of the cart is given by the sum of the cart mass and the mass of the additional weights that are added to balance the weight of the pendulum attached to the cart. The movement of the cart is constrained only in horizontal direction whereas the pendulum can rotate in the x-y plane. Hence the system can be represented by the two state variables namely, the horizontal displacement of the cart and the angular displacement of the pendulum. The coulomb's frictional force exerted by the cart pinion arrangement and the force on the cart due to pendulum's action are assumed to be negligible for the modeling of the system. The Cartesian co-ordinates of the cart-inverted pendulum is represented as shown in Fig. 2. The global frames are fixed as $x - y$ and the position of the pendulum with respect to the global frame is given by $x_p - y_p$corresponding to the x and y global reference frame. The mathematical model of the setup shown in Fig. 2 is obtained by applying the Euler-Lagrangian energy equation.

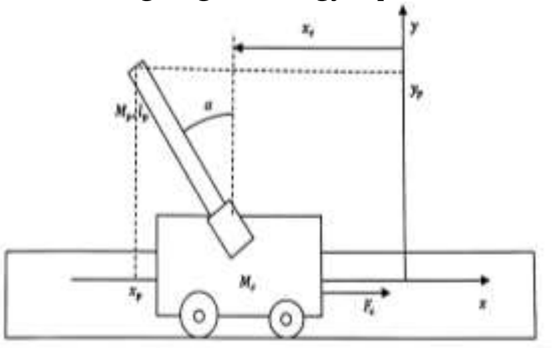

Fig. 2. Cartesian co-ordinates of Cart-Inverted Pendulum System

B. *Euler-Lagrangian Formulation*

The Lagrangian formulation is based on the differentiation of the energy terms with respect to the system's state variables and time. When the complexity of the system increases, the Lagrangian method becomes relatively simpler to use. Lagrangian method is based on the following two generalized equations: one for the linear motions and the other for the rotational motions. Because of the effectiveness, the Lagrangian method is used for modeling the complex systems which have translational as well as rotational motions. The Lagrangian is defined as

$$L = K - P \qquad (1)$$

where L is the lagrangian, K is the total kinetic energy of the system and P is the total potential energy of the system.

The Euler-Lagrangian for the cart-inverted pendulum system is given by

$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_c} - \frac{\partial L}{\partial x_c} \quad (2)$$

$$T_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} \quad (3)$$

where F_i and $T_i$are the force applied on the coordinate x_c and α respectively. On substitution, the lagrangian is thus expressed as shown below

$$L = \frac{1}{2}\left(M_p + M_C\right)\dot{x}_c^2(t) - M_p l_p\, cos\big(\alpha(t)\big)\dot{\alpha}(t)\dot{x}_c(t) + \frac{1}{2}\left(l_p + M_p I_p^2\right)\dot{\alpha}^2(t) - P \quad (4)$$

The nonlinear equations of motion are obtained as

$$F_c(t) = \left(M_p + M_c\right)\ddot{x}_c(t) + B_{eq}\dot{x}_c(t) + M_p l_p \cos\big(\alpha(t)\big)\ddot{\alpha}(t) - M_p l_p \sin\big(\alpha(t)\big)\dot{\alpha}^2(t) \quad (5)$$

$$-M_p l_p \cos\big(\alpha(t)\big)\dot{\alpha}(t)\ddot{x}_c(t) + \left(l_p + M_p I_p^2\right)\ddot{\alpha}(t) + B_p\dot{\alpha}(t) - M_p g l_p \sin\big(\alpha(t)\big) = 0 \quad (6)$$

C. *Model Linearization*

The nonlinear model is linearized around the equilibrium point i.e. upright position such that $\sin(\alpha) \cong \alpha, \cos(\alpha) \cong 1$. The linearized model is written in the state space form as

$$\dot{x} = Ax + Bu \quad (7)$$

$$y = Cx \quad (8)$$

The state space model of the system is thus obtained as

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{g M_p^2 I_p^2}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} & \dfrac{-B_{eq} M_p I_P^2}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} & \dfrac{-M_p I_p B_p}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} \\ 0 & \dfrac{M_p g I_p\left(M_p + M_c\right)}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} & \dfrac{-M_p I_p B_{eq}}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} & \dfrac{-\left(M_p + M_c\right)B_p}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{I_p + M_p I_p^2}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} \\ \dfrac{M_p I_p}{\left(M_p + M_c\right)I_p + M_p M_c I_P^2} \end{bmatrix}$$

The cart-pendulum system parameters that are used to obtain the state space model are shown in Table.1.

Table.1 System Parameters of Cart-Inverted Pendulum system

Parameter	**Description**	**Value**
M_c	Mass of the cart	1.0731 kg
M_p	Mass of the pendulum	0.127kg
l_p	Length of the pendulum from center to center of gravity	0.1778 m
I_p	Moment of inertia of the pendulum	$1.2 \times 10^{-3} kgm^2$
g	*Acceleration due to gravity*	*9.81* m/sec^2
B_p	Viscous damping co-efficient at pendulum axis	0.0024 $N\ m\ sec/rad$
B_{eq}	Viscous damping co-efficient at motor pinion	5.4 $N\ m\ sec/rad$

By substituting the parameters given in the Table. 1 in A and B matrices, the state space model of the system is obtained as shown below,

$$\begin{bmatrix} \dot{x}_c \\ \dot{\alpha} \\ \ddot{x}_c \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.8703 & -4.8987 & -0.0094 \\ 0 & 46.2580 & -21.2169 & -0.5012 \end{bmatrix} \begin{bmatrix} x_c \\ \alpha \\ \dot{x}_c \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.9072 \\ 3.9291 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ \alpha \\ \dot{x}_c \\ \dot{\alpha} \end{bmatrix}$$

D. *Stability of Cart Pendulum System*

The cart-pendulum system is an unstable system in open loop and this can be verified with the pole zero plot for the obtained state space model. The pole zero plot for the system is shown in Fig. 3. As seen from the pole zero map, the poles of the obtained state space model are located at 0, -5.6041, -0.142, 5.5651. This implies that the system inherently is not stable and thus needs a proper controller to be designed that can bring all the closed loop poles to the left half of the s-plane making the system stable. The need for proper controller design is thus clearly visible and that system would be unstable in open loop.

Hence the design of controller gain is significance for such systems. If the controller designed works well for the cart-pendulum system, then they can be implemented in a wide variety of real time applications. The controller designed for such systems find their application in aerospace systems which are much more complex and unstable. Also, they can be used to control similar under actuated system as well.

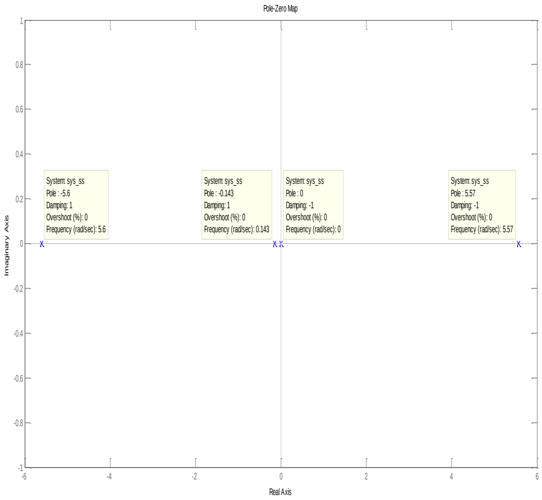

Fig. 3. Pole zero map of the cart-pendulum system model

Design of Robust LQR Control using ADP

The basic idea of Linear Quadratic Regulator (LQR) controller is to solve the weighting matrices selection problem. One of the important challenges in the design of LQR for real time applications is the optimal choice of state matrix (Q) and input weighting matrix (R), which play a vital role in determining the performance and optimality of the controller. Commonly, trial and error approach is employed for selecting the weighting matrices, which is not only tedious but also time consuming and results in non-optimal response. Hence, to choose the elements of Q and R matrices optimally, an adaptive dynamic programming (ADP) algorithm is formulated and applied for minimizing the performance index or the cost function. Moreover, by minimizing a quadratic cost function which consists of two penalty matrices (Q and R), LQR yields an optimal response between the control input and speed of response. Hence, the LQR techniques have been successfully applied to a large number of complex systems such as vibration control system, fuel cell system and aircraft.

In order to obtain the Q and R matrices optimally, iteration is performed using adaptive dynamic programming. The value of Q and R for which the cost function tends to be minimum is considered to be the optimal values of the matrices Q and R. The block diagram for the LQR controller using adaptive dynamic programming is shown in Fig. 4.

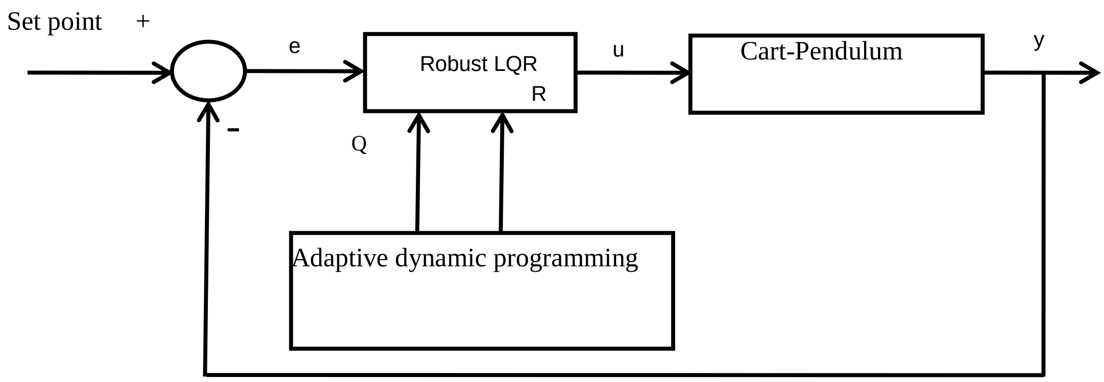

Fig. 4. Block diagram of LQR controller for cart pendulum system using ADP

Consider a linear time invariant system (LTI),

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (9)$$
$$y = Cx(t) \quad (10)$$

where $x(t)$ is the state vector and $u(t)$ is the input vector, determine the matrix $K \in R^{n\times m}$ such that the static, full state feedback control law,

$$u = -K\,x(t) \quad (11)$$

satisfies the following criteria,

a) the closed-loop system is asymptotically stable
b) the quadratic performance functional and the cost function

$$J(K) = \frac{1}{2}\int_0^{\infty}[x^T(t)\,Q\,x(t) + u^T(t)R\,u(t)]dt \quad (12)$$

is minimized. Q is a nonnegative definite matrix that penalizes the departure of system states from the equilibrium, and R is a positive definite matrix that penalizes the control input.

The following are the steps to design LQR control where Q and R values are selected by iteration method:

Step: 1: Solve the matrix Algebraic Riccati Equation (ARE)

$$-PA - A^TP - Q + PBR^{-1}B^TP = 0 \quad (13)$$

Step: 2: Determine the optimal state $x^*(t)$ from

$$\dot{x}^*(t) = [A - BR^{-1}B^TP]\,x^*(t) \quad (14)$$

Step: 3: Obtain the optimal control $u^*(t)$ from

$$u^*(t) = -R^{-1}B^TPx^*(t) \quad (15)$$

Step: 4: Obtain the optimal performance index from

$$J^* = \frac{1}{2}{x^*}^T(t)\,P\,x(t) \quad (16)$$

Step: 5: Iterate the Q and R values from 0 to n, where n represents the number of iteration to be performed till the performance index or cost function gets minimized.

The weighting matrices Q and R are important components of an LQR optimization process. The composition of Q and R elements has great influences on system performance. The designer need not to worry about the choice of Q and R values as it can be resolved using iteration method. The optimal control problem,

$$u^*(t) = arg\,\min_{u(t)} V(t_0, x(t_0), u(t)) \quad (17)$$

and the infinite horizon quadratic cost function to be minimized is expressed as

$$V(x(t_0), t_0) = \int_{t_0}^{\infty} \begin{pmatrix} x^T(\tau)\, Q\, x(\tau) \\ + \\ u^T(\tau) R\, u(\tau) \end{pmatrix} d\tau \qquad (18)$$

with $Q \geq 0, R > 0$ and $(Q^{\frac{1}{2}}, A)$ detectable.

The solution of this optimal control problem, determined by Bellman's optimality principle is given by

$$u(t) = -Kx(t) \qquad (19)$$

$$K = R^{-1}B^T P \qquad (20)$$

where the matrix P is the unique positive definite solution of the Algebraic Riccati Equation (ARE),

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (21)$$

Under the detectability condition for $(Q^{\frac{1}{2}}, A)$, the unique positive semi-definite solution of the ARE determines a stabilizing closed loop controller given by equation (20). It is important to note that, in order to solve equation (21), complete knowledge of the model of the system is needed i.e., both the system matrix A and control input matrix B must be known.

2. RESULTS AND DISCUSSIONS

This section focuses on the selection of weighting matrices Q and R by an iterative approach using adaptive dynamic programming (ADP) which makes the design of LQR to be robust. The value of Q and R for which the cost function defined becomes minimum, is chosen as the optimum solution to the problem. This method replaces the trail and error method where Q and R matrices are selected by the experience of the user. The following sections illustrate the results obtained using MATLAB platform.

E. *Open Loop Response of the Inverted Pendulum System*

The fig.5 shows the open loop response of the inverted pendulum. From the response, it is clear that the system is unstable since the output is unbounded.

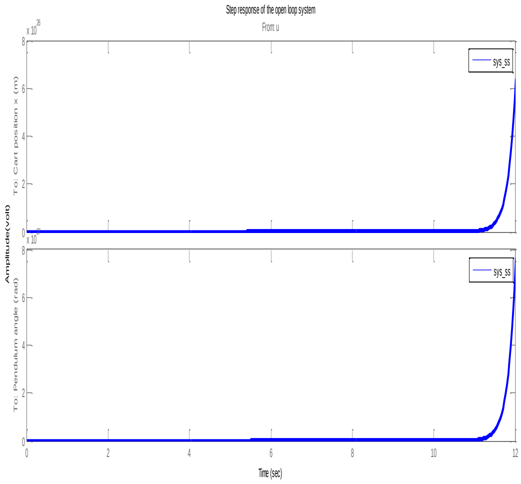

Fig.5. Step response of the open-loop system

F. Closed Loop Response of the Inverted Pendulum System

The fig.6 shows the pole zero plot of the closed loop system which is obtained using LQR control. It can be seen from the fig.6 that the poles are located at the left-half of the plane. This indicates that the system become stable using LQR control.

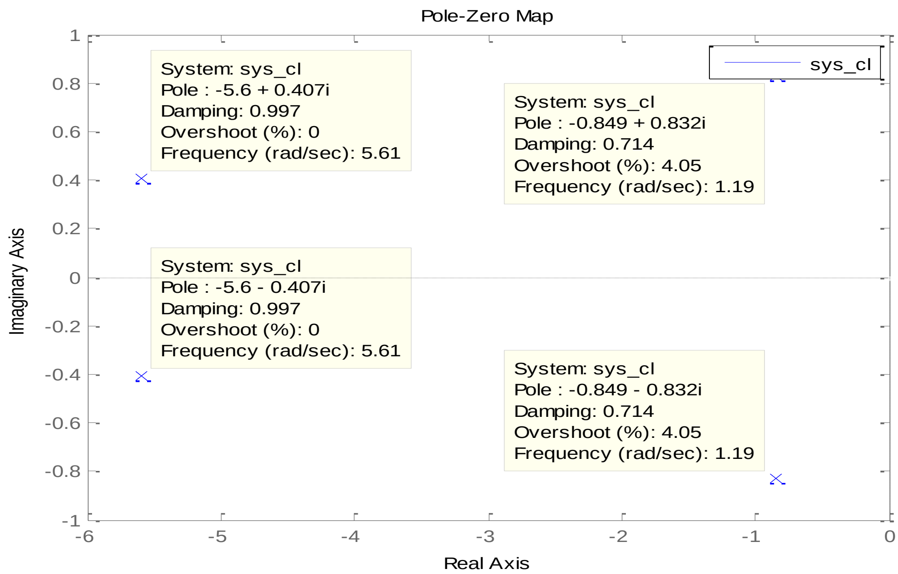

Fig.6. Pole-Zero map of the closed-loop system

The fig.7 shows the closed loop response of the inverted pendulum system and it can be seen from the figure that both the cart position $(x = 1m)$ and pendulum angle$(\alpha = 0\ or\ 180^{\circ})$ attain their desired value.

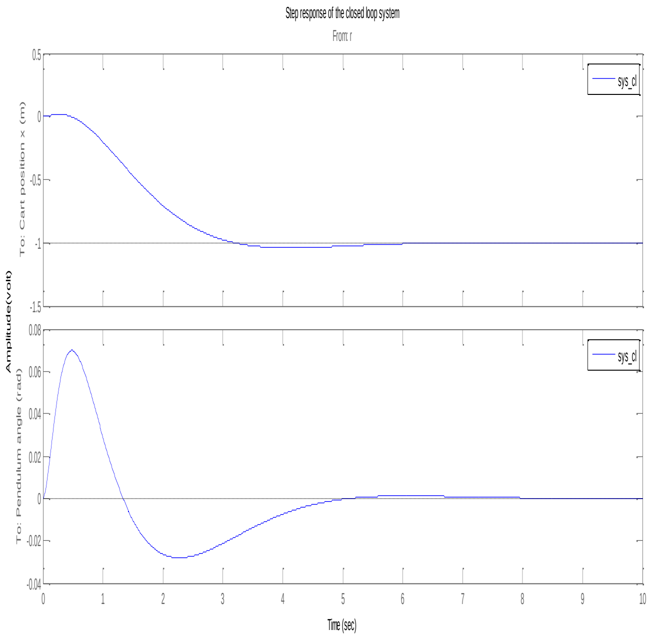

Fig.7. Step response of the closed-loop system

G. Response of Inverted Pendulum System using LQR Control

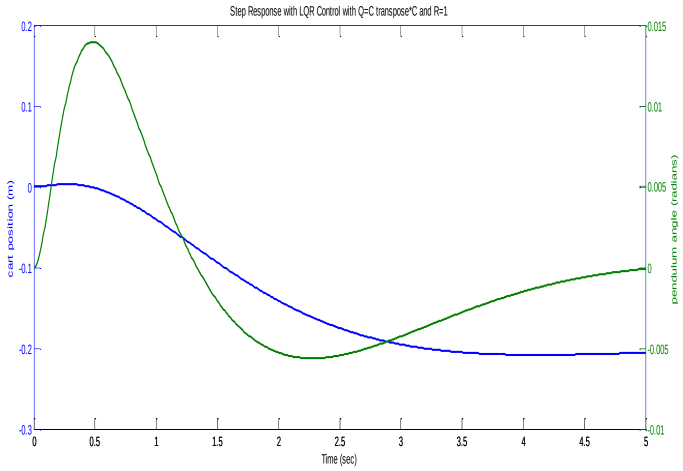

Fig.8. Step response with LQR control

It is shown from the fig.8 that the Q and R values are given ($Q = C' * C$ and $R = 1$). The cart position and pendulum angle settle to its desired value but settling time is more (4 seconds) and rise time is less (0.5 seconds).

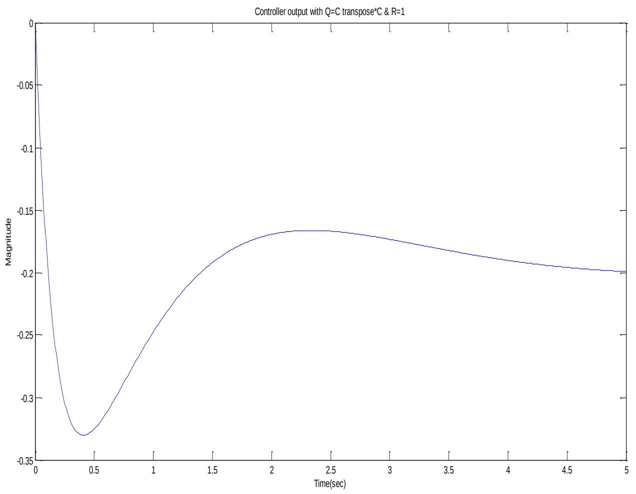

Fig.9. Controller output using LQR control

The fig.9 shows the controller output obtained using LQR control and it implies that the response is doesn’t settles properly.

H. *Response of the Inverted Pendulum System with LQR Control using ADP*

The step response of the inverted pendulum system with LQR control using ADP is shown in fig.10 where the Q and R values are obtained by iteration. Here the rise time is lesser (0.25 seconds) than the step response obtained using LQR control. Also the settling time (0.7 seconds) of both cart position and pendulum angle with LQR control using ADP is lesser than the settling time of LQR control.

From this, the most optimum values of Q and R are chosen iteratively and also the response is improved in this method when compared with the response of conventional LQR control. The fig.11 shows the controller response of LQR control using ADP whose rise time and settling time is lesser than the controller response obtained by using conventional LQR control.

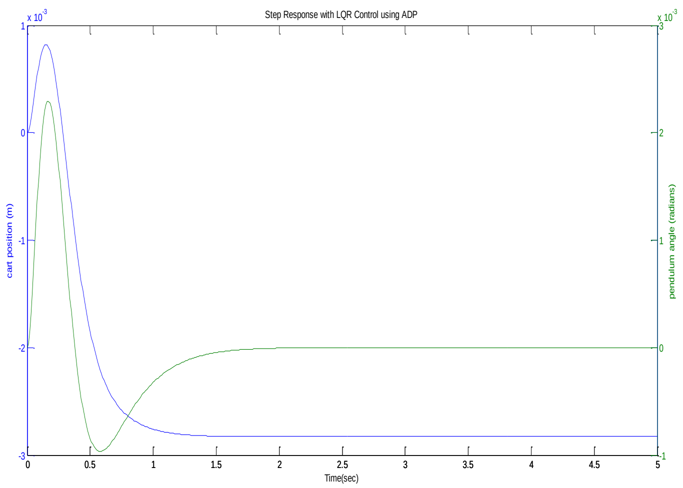

Fig.10. Step response with LQR using ADP

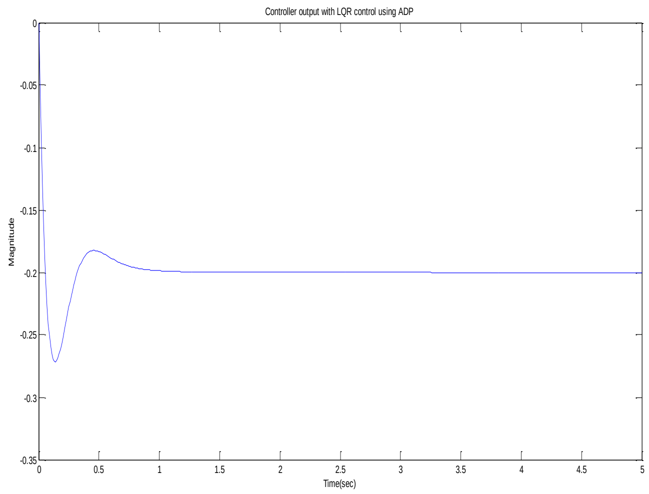

Fig.11. Controller output with LQR control using ADP

Table.2 Comparative analysis - step response and controller response of LQR control and LQR control using ADP

		LQR Control	**RLQR Control**
Time Response (in seconds)	**Parameters**	**Step Response**	
Rise Time	**Cart Position (in meters)**	0.356	0.276
	Pendulum Angle (in radians)	0.498	0.295
Settling Time	**Cart Position (in meters)**	3.676	1.491
	Pendulum Angle (in radians)	4.983	1.897
		Controller Response	
	Controller Output	4.568	1.282

3. CONCLUSION

In this paper a robust linear quadratic regulator (RLQR) control has been introduced for nonlinear systems. The choice of weighting matrices Q and R plays a major role when performance index or cost function is taken into account. Comparative analysis were also made between the conventional LQR control and RLQR control as shown in Table.2 which strengthens the objective of this project. To avoid the conventional trial and error method, an iterative method called Adaptive Dynamic Programming (ADP) is introduced. The term robust here means that the Q and R values obtained in this method are the optimal values and replaces the conventional LQR control into the robust LQR control. This method reduces the time required to select the weighting matrices which are being chosen from user's previous experience. The value of Q and R is selected for which it results in minimum cost function. The iteration will run for 'n' number of times till the performance index become minimum. This project can be extended for discrete-time nonlinear system as future work. As mentioned earlier, there are several types of adaptive dynamic programming which can be selected based on the requirements or constraints.

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