International Journal of Aquatic Science
ISSN: 2008-8019
Vol 12, Issue 02, 2021

Results on Controllability of Volterra Integro-Dynamic Matrix Sylvester Impulsive Systems on Time Scales

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***Abstract*: In this paper, we derive results on controllability of Volterra intrgro-dynamical matrix Sylvester impulsive linear and non-linear systems on time scales by using Banach fixed point theorem**.

Keywords: **controllability, Matrix impulsive system, time scale calculus, fixed point theorem.**

1. INTRODUCTION:

Sudden changes in their states characterize many physical problems. These sudden changes are known as impulsive effective in the system. Integro-differential equations with impulsive matrix dynamical systems have been considered necessary in varied applications like physics, biological systems such as heartbeats, economics, mechanical system with impact, control theory, etc. in 1988, Hilger introduced the calculus on time scales in his Ph.D. thesis. The study of dynamic equations encapsulates both the continuous as well as discrete analysis of the system. In 1960, Kalman (1960) was the first person who introduced the concept of controllability that formed the backbone of the modern control theory, roughly speaking, a system is known as controllable if, by using control input, it can be driven from a given initial state to desired final form within the finite time. The Lyapunov type system is helpful for many branches of science and engineering systems at control theory.
The principal motivation behind this paper, the results on controllability of the solution of delta differentiable time-varying and time-invariant Volterra integro-dynamical matrix Sylvester impulsive systems are given by

$$\begin{cases} Y^{\Delta}(t) = A(t)Y(t) + Y(t)B(t) + C(t)U(t) + \int_0^t \big(T_1(t,s)Y(s) + Y(s)T_2(t,s)\big)\Delta s \\ \qquad\qquad + h(t, Y(t)), \quad t \in \mathbb{T}_0 \backslash \{t_k\}_{k=1}^{\infty} \\ Y(t_k^+) = (I + D_k)Y(t_k), \quad t = t_k, k = 1, 2, \dots, m \qquad (1.1) \\ Y(t_0) = Y_0 \end{cases}$$

Where $\mathbb{T}$ has the property unbounded above time scale with bounded graininess, $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}, t_k \in \mathbb{T}_0$ are right dense. $D_k \in M_n(\mathbb{R})$, $A(t), B(t), C(t), T_1(t)$ and $T_2(t)$ with dimensions $n \times n$, $n \times n, n \times n, n \times m$ and $n \times n$ rd-continuous matrices respectively. $h: I \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous on $\mathbb{T}_0$. $Y(t) \in \mathbb{R}^n$ is state variable.$U(t) \in \mathbb{R}^{m\times n}$ is the control input. The generalized Delta derivative of Y is $Y^{\Delta}(t)$.
In the second section, we analyse some standard properties of time scale calculus also derive basic concepts for converting a given matrix-valued system into a Kronecker product system by using a variation of parameters. In the third section, we derive results on the controllability for linear and non-linear of the Volterra integro-dynamical matrix Sylvester impulsive system on time scales.

PRELIMINARIES

We recollection some fundamental definitions, notations and useful lemmas.

Definition 2.1[6] A nonempty closed subset of $\mathbb{R}$ is called a time scale. It is denoted by $\mathbb{T}$. We define a $\mathbb{T}$ interval as $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T}: a \le t \le b\}$, accordingly, we define $(a,b)_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}, (a,b]_{\mathbb{T}}$and so on. Also, we define $\mathbb{T}^k = \mathbb{T}\{max\mathbb{T}\}$ if max $\mathbb{T}$ exists, otherwise the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = inf\{s \in \mathbb{T}: s > t\} \in \mathbb{T}$ with the substitution $\inf\{\emptyset\} = sup\mathbb{T}$ and The graininess function $\mu(t): \mathbb{T} \to [0, \infty)$ by $\mu(t) = \sigma(t) - t, for\ all\ t \in \mathbb{T}$.

Definition 2.2[6] Let $Y: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. the delta derivative of $Y^{\Delta}(t)$ is the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighbourhood U of τ such that

$$|[Y(\sigma(\tau)) - Y(s)] - Y^{\Delta}(\tau)[\sigma(\tau) - s]| \le \varepsilon|\sigma(\tau) - s|, \text{ for all } s \in U$$

Definition 2.3[6] If $F: \mathbb{T}^k \to \mathbb{R}$ is said to be anti-derivative of $f: \mathbb{T}^k \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ fulfilled, for all $t \in \mathbb{T}^k$.Then

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

Definition 2.4 [7] A function $Y: \mathbb{T} \to \mathbb{R}$ is called the regressive if $1 + \mu(t)A(t) \neq 0\ \ \forall t \in \mathbb{T}$ and the set of all regressive functions are denoted by $\mathcal{R}$.Also,x is called positive regressive function if $1 + \mu(t)A(t) > 0\ \forall$ t $\in \mathbb{T}$ and it is denoted by $\mathcal{R}^+$.

Definition 2.5 [4] The right dense continuous matrices are X and Y on time scale $\mathbb{T}$, then

$$(X \otimes Y)^{\Delta}(t) \ = X^{\Delta}(t) \otimes Y(t) \ + \ X(\sigma(t)) \otimes Y^{\Delta}(t)$$

We put the vec operator to the equation (1.1), then it is converted into a Kronecker product dynamical system by using Kronecker product properties [4], we have

$$\begin{cases} z^{\Delta}(t) = P(t)z(t) + Q(t)\widehat{U}(t) + \int_0^t G(t,s)z(s)\Delta s + H\big(t, z(t)\big),\ t \in \cup_{k=0}^{m} (s_k, t_{k+1}]_{\mathbb{T}}, \\ z(t_k^+) = [I_n \otimes R_k]z(t_k),\ \ t \in (t_k, s_k]_{\mathbb{T}},\ k = 1, 2, \ldots, m, \\ z(t_0) = z_0. \end{cases} \quad (2.1)$$

Where $\mathbb{T}$ is a time scale with $s_k, t_k \in \mathbb{T}$ are rd points with $0 = s_0 = t_0 < t_1 < s_1 < t_2 < \cdots s_m < t_{m+1} = T, z(t_k^+) = \lim_{h\to 0^+0} z(t_k + h), z(t_k^-) = \lim_{h\to 0^+0} z(t_k - h)$,

it is representing the right and left limits of x(t) at t=$t_k$in the sense of $\mathbb{T}$, $z \in \mathbb{R}^{n^2}$is a state variable. $R_k = (I + D_k))] \in C_{rd}\mathcal{R}(M_{n^2 x n^2}(\mathbb{R}))$
$P(t) = [B^* \otimes I + I \otimes A] \in C_{rd}\mathcal{R}\big(M_{n^2 x n^2}(\mathbb{R})\big)$, and $Q(t) = (I \otimes C) \in C_{rd}\mathcal{R}\big(M_{n^2 x n^2}(\mathbb{R})\big)$
$G(t,s) = [(T_2 \otimes I_n) + (I_n \otimes T_1)] \in C_{rd}\mathcal{R}(M_{n^2 x n^2}(\mathbb{R}))$.
$\widehat{U}(t) = VevU(t) \in C_{rd}\mathcal{R}\big(M_{n^2 \times 1}(\mathbb{R})\big)$ and $H(t) = Vech(t) \in C_{rd}\mathcal{R}\big(M_{n^2 \times 1}(\mathbb{R})\big)$.

Definition2.7[7] Let $P, Q \in \mathcal{R}$, then

i. $P \ominus Q = P \oplus (\ominus Q)$

ii. $P \oplus B = P + Q + \mu(\tau)PQ$

iii. $\ominus P = \frac{-P}{1+\mu(\tau)P}$

Lemma2.2: [6] if $z \in PC_{rd}(\mathbb{T}, \mathbb{R}^+)$ satisfies the inequality condition. Then

$$z(t) \leq \alpha + \int_a^t P(s)z(s)\Delta s + \sum_{a<t_k<t} \beta_k z(t_k), \forall t \in \mathbb{T},$$

Then

$$z(t) \leq \alpha \prod_{a<t_k<t} (1 + \beta_k)e_P(t,a), \forall t \in \mathbb{T}.$$

Lemma2.1[7] If $P \in \mathcal{R}$, then

i. $(e_{\ominus P}(t,s))^\Delta = \ominus P(t)e_{\ominus P}(t,s)$.

ii. $e_0(t,s) = 1 \; and \; e_P(t,t) = 1$.

iii. $e_P(t,s) = \frac{1}{e_P(s,t)} = e_{\ominus P}(s,t)$

iv. $e_P(\sigma(t), s) = \left(1 + \mu(\tau)P(t)\right)e_P(t,s)$.

v. $e_P(t,s)e_P(s,r) = e_P(t,r)$.

Definition2.6[7] For $P \in \mathcal{R}$, the generalized exponential function on $\mathbb{T}$ is defined as

$$e_P(t,s) = exp\left(\int_s^t \xi_{\mu(t)}\left(P(\tau)\right)\Delta\tau\right), \quad t,s \in \mathbb{T},$$

Where $\xi_{\mu(t)}(P(\tau)) = \begin{cases} \frac{Log(1+\mu(\tau)P,}{\mu(\tau)} & if \; \mu(\tau) \neq 0. \\ P & if \; \mu(\tau) = 0. \end{cases}$ is a cylinder transformation.

Equivalent to the system (2.1), consider the following linear Volterra integro-dynamical matrix Sylvester without impulsive:

$$\begin{cases} z^\Delta(t) = P(t)z(t) + \int_0^t G(t,s)z(s)\Delta s, & t \in I, \\ z(t_0) = z_0 \end{cases} \qquad (2.2)$$

An $n^2 \times n^2$ matrix is defined to be a real-valued function of $\emptyset(t,s)$ and it is denoted by

$\emptyset(t,s) = [z_1(t,s), z_2(t,s), \dots, z_{n^2}(t,s),]$,

where $z_k(t,s), k = 1,2,3, \dots, n^2$ are n^2 linearly independent solution of the system (2.2). the principal matrix $\emptyset(t,s)$ is known as the transition matrix if $\emptyset(t,0) = I_{n^2 \times n^2} \; at \; t = 0$. If $z(t) = \emptyset(t,0)z_0$ is a unique solution of the system (2.2).

Lemma2.3: Let $\emptyset(t,s)$ be the transition matrix of the system (2.2), then

1. $\emptyset(t,\tau) = \emptyset(t,s)\emptyset^{-1}(\tau,s), \emptyset^{-1}(t,\tau) = \emptyset(\tau,t)$;
2. $\emptyset^{\Delta_t}(t,s) = P(t)\emptyset(t,s) + \int_s^t G(t,\tau)\emptyset(\tau,s)\Delta\tau$;
3. $\emptyset^{\Delta_s}(t,s) = -\emptyset\left(t,\sigma(s)\right)P(s) - \int_{\sigma(s)}^t G(t,\sigma(\tau))\emptyset(\tau,s)\Delta\tau$.

CONTROLLABILITY

3.1. Controllability of linear impulsive system

We consider non-linear system (2.1) to linear impulsive system, we have

$$\begin{cases} z^{\Delta}(t) = P(t)z(t) + Q(t)\widehat{U}(t) + \int_0^t G(t,s)z(s)\Delta s, & t \in \cup_{i=0}^{m} (s_k, t_{k+1}]_{\mathbb{T}}, \\ z(t_k^+) = [I_n \otimes R_k]z(t_k), & t \in (t_k, s_k]_{\mathbb{T}}, k = 1, 2, 3, \dots, m, \\ z(t_0) = z_0. \end{cases} \quad (3.1)$$

Definition 3.1 A function $z \in PC(I, \mathbb{R}^{n^2})$ is known as the solution of the system (3.1) is z(t) satisfies $z(t_0) = z_0, z(t_k^+) = [I_n \otimes R_k]z(t_k), \ \forall \ t \in (t_k, s_k]_{\mathbb{T}}$, k = 1, 2, 3, ..., m, and

$$z(t) = \emptyset(t,0)z_0 + \int_0^t \emptyset\big(t,\sigma(\tau)\big)Q(\tau)\widehat{U}(\tau)\Delta\tau, \forall\, t \in [0,t_1]_{\mathbb{T}}. \quad (3.2)$$

$$z(t) = \emptyset(t,s_K)[I_n \otimes R_k]z(t_k) + \int_{s_K}^t \emptyset\big(t,\sigma(\tau)\big)Q(\tau)\widehat{U}(\tau)\Delta\tau, \forall\, t \in (s_k, t_{k+1}]_{\mathbb{T}}, k = 1, 2, 3, \dots, m, \quad (3.3)$$

Remarks 3.1[19] it is apparent that if there is no impulsive condition, thus, the system (3.1), we have

$$\begin{cases} z^{\Delta}(t) = P(t)z(t) + Q(t)\widehat{U}(t) + \int_0^t G(t,s)z(s)\Delta s, & t \in I, \\ z(t_0) = z_0, \end{cases} \quad (3.4)$$

and the solution of the system (3.4) is

$$z(t) = \emptyset(t,0)z_0 + \int_0^t \emptyset\big(t,\sigma(s)\big)Q(s)\widehat{U}(s)\Delta s.$$

The controllability results for (3.4).

Definition 3.2. The linear system (3.4) is called controllable on $[t_0, T]$ if for every $z_0, z_T \in \mathbb{R}^{n^2}, \exists$ a piecewise rd-continuous input signal $\widehat{U}(.) : [0,T] \to \mathbb{R}^{n^2}$ such that the corresponding solution of (2.1) satisfies $z(t_0) = z_0$ and $z(T) = z_T$.

Definition 3.3. The matrix Sylvester impulsive system (3.1) is called controllable on $[0,T]$ if for k=1, 2, …, m, it is controllable on $[0,t_1]_{\mathbb{T}}$ and $[s_k, t_{k+1}]_{\mathbb{T}}$.

The corresponding Grammian matrices:

$$\mathcal{W}_0(0,t_1) = \int_0^{t_1} \emptyset\big(t_1,\sigma(\tau)\big)Q(\tau)Q^*(\tau)\emptyset^*\big(t_1,\sigma(\tau)\big)\Delta\tau \quad (3.5)$$

$$\mathcal{W}_k(s_k,t_{k+1}) = \int_{s_k}^{t_{k+1}} \emptyset\big(t_{k+1},\sigma(\tau)\big)Q(\tau)Q^*(\tau)\emptyset^*\big(t_{k+1},\sigma(\tau)\big)\Delta\tau, k = 1,2,\dots,m \quad (3.6)$$

Where Q^* denotes the transpose of the matrix Q.

Lemma 3.1 The linear matrix Sylvester impulsive system (3.1) is controllable on $[0,t_1]_{\mathbb{T}}$, if and if the matrix $\mathcal{W}_0(0,t_1)$ given by the system (3.6) is invertible.

Proof: we suppose that the matrix $\mathcal{W}_0(0,t_1)$ is invertible. Thus, we can define the input control $\widehat{U}(t)$ as

$$\widehat{U}(t) = Q^*(\tau)\emptyset^*\big(t_1,\sigma(\tau)\big)\mathcal{W}_0^{-1}(0,t_1)\big(z_{t_1} - \emptyset(t_1,0)z_0\big), \forall\, t \in [0,t_1]_{\mathbb{T}} \quad (3.7)$$

Now, put $t = t_1$, in the solution (3.2) of the system (3.1), we get

$$z(t_1) = \emptyset(t_1,0)z_0 + \int_0^{t_1} \emptyset\big(t_1,\sigma(\tau)\big)Q(\tau)\ \widehat{U}(\tau)\ \Delta\tau$$

$$= \emptyset(t_1,0)z_0 + \int_0^{t_1} \emptyset\big(t_1,\sigma(\tau)\big)Q(\tau)\ Q^*(\tau)\emptyset^*\big(t_1,\sigma(\tau)\big)\mathcal{W}_0^{-1}(0,t_1)\big(z_{t_1} - \emptyset(t_1,0)z_0\big)\ \Delta\tau$$

$$= \emptyset(t_1,0)z_0 + \mathcal{W}_0(0,t_1)\mathcal{W}_0^{-1}(0,t_1)\big(z_{t_1} - \emptyset(t_1,0)z_0\big)$$

$$= z_{t_1}.$$

Hence, input control (3.13) is right for $t \in [0,t_1]_{\mathbb{T}}$, Therefore, the linear matrix Sylvester impulsive system (3.1) is controllable on $[0,t_1]_{\mathbb{T}}$.
Conversely, we suppose that the system (3.1) is controllable on $[0,t_1]_{\mathbb{T}}$. Now, we suppose that the $\mathcal{W}_0(0,t_1)$ is not invertible matrix. Since, there exists a non-zero vector $z_\alpha \in \mathbb{R}^{n^2}$ such that
$z_\alpha{}^*\mathcal{W}_0(0,t_1)z_\alpha = 0.$
Now, put the value of $\mathcal{W}_0(0,t_1)$, in the above system, we have

$$\int_0^{t_1} z_\alpha{}^*\emptyset\big(t_1,\sigma(\tau)\big)Q(\tau)\;Q^*(\tau)\emptyset^*\big(t_1,\sigma(\tau)\big)z_\alpha\,\Delta\tau = 0.$$

Therefor,

$$z_\alpha{}^*\emptyset\big(t_1,\sigma(\tau)\big)Q(\tau) = 0, \tau \in [0,t_1]_{\mathbb{T}}. \qquad (3.8)$$

Since, the system (3.1) is controllable on $[0,t_1]_{\mathbb{T}}$, so if we can indicate
$z_0 = \emptyset(0,t_1)\big(z_\alpha + z_{t_1}\big),$
Thus, there exists a piecewise rd-continuous input control $\widehat{U}(.)$ such that

$$z_{t_1} = \emptyset(t_1,0)\left(\emptyset(0,t_1)\big(z_\alpha + z_{t_1}\big)\right) + \int_0^{t_1}\emptyset\big(t_1,\sigma(\tau)\big)Q(\tau)\;\widehat{U}(\tau)\,\Delta\tau,$$

Which gives $z_\alpha{}^*z_\alpha = 0$. Thus, it is contradiction $z_\alpha \neq 0$, and, hence, the matrix $\mathcal{W}_0(0,t_1)$ is invertible.

Lemma 3.2 The linear matrix Sylvester impulsive system (3.1) is controllable on $[s_k,t_{k+1}]_{\mathbb{T}}, k = 1,2,\dots,m,$ with $t_{m+1} = T$ if and only if the matrices $\mathcal{W}_k(s_k,t_{k+1}), k = 1,2,\dots,m$ given by (3.7) are invertible.

Proof: Since, the matrices $\mathcal{W}_k(s_k,t_{k+1})$ are invertible, then we can define the input control $\widehat{U}(t)$ as

$$\widehat{U}(t) = Q^*(\tau)\emptyset^*\big(t_{k+1},\sigma(\tau)\big)\mathcal{W}_k{}^{-1}(s_k,t_{k+1})\left(z_{t_{k+1}} - \emptyset(t_{k+1},s_k)[I_n\otimes R_k]z(t_k)\right),$$
$$\forall t \in (s_k,t_{k+1}]_{\mathbb{T}}, k = 1,2,\dots,m, \qquad (3.9)$$

Now, put $t = t_{k+1}$,in the solution (3.3) of the system (3.1), we have

$$z(t_{k+1}) = \emptyset(t_{k+1},s_k)[I_n\otimes R_k]z(t_k) + \int_{s_k}^{t_{k+1}}\emptyset\big(t_{k+1},\sigma(\tau)\big)Q(\tau)\;\widehat{U}(\tau)\Delta\tau$$
$$= \emptyset(t_{k+1},s_k)[I_n\otimes R_k]z(t_k)$$
$$+ \int_{s_k}^{t_{k+1}}\emptyset\big(t_{k+1},\sigma(\tau)\big)Q(\tau)\;Q^*(\tau)\emptyset^*\big(t_{k+1},\sigma(\tau)\big)\mathcal{W}_k{}^{-1}(s_k,t_{k+1})$$
$$\times\left(z_{t_{k+1}} - \emptyset(t_{k+1},s_k)[I_n\otimes R_k]z(t_k)\right)\Delta\tau$$
$$= \emptyset(t_{k+1},s_k)[I_n\otimes R_k]z(t_k)$$
$$+ \mathcal{W}_k(s_k,t_{k+1})\mathcal{W}_k{}^{-1}(s_k,t_{k+1})\left(z_{t_{k+1}} - \emptyset(t_{k+1},s_k)[I_n\otimes R_k]z(t_k)\right)$$
$$= z_{t_{k+1}}.$$

Therefore, control input (3.9) is right for $t \in (s_k,t_{k+1}]_{\mathbb{T}}$. Hence, the linear matrix Sylvester impulsive system (3.1) is controllable on $(s_k,t_{k+1}]_{\mathbb{T}}$.
Conversely, we suppose that the system (3.1) is controllable on $[s_k,t_{k+1}]_{\mathbb{T}}$. then, we suppose that for k=1, 2, …, m, the matrix $\mathcal{W}_k(s_k,t_{k+1})$ are not invertible. Therefore, there exist a non-zero vectors $z_\alpha \in \mathbb{R}^{n^2}$, k = 1, 2, …, m, such that
$z^*_{\alpha_k}\mathcal{W}_k(s_k,t_{k+1})z_{\alpha_k} = 0.$
Now, put the value of $\mathcal{W}_k(s_k,t_{k+1})$, in the above equation, we have

$$\int_{s_k}^{t_{k+1}} z_{\alpha_k}^* \emptyset(t_1,\sigma(\tau))Q(\tau)\ Q^*(\tau)\emptyset^*(t_1,\sigma(\tau))\, z_{\alpha_k}\Delta\tau = 0.$$

Therefore,

$$z_{\alpha_k}^* \emptyset(t_{k+1},\sigma(\tau))Q(\tau) = 0, \quad \tau \in (s_k, t_{k+1}]_{\mathbb{T}}, k = 1,2,\dots,m \qquad (3.10)$$

Since, the system (3.1) is complete controllable on $[s_k, t_{k+1}]_{\mathbb{T}}$, so if we can indicate $[I_n \otimes R_k]z(t_k) = \emptyset(s_k, t_{k+1})(z_{\alpha_k} + z_{\alpha_{k+1}})$ in $[s_k, t_{k+1}]_{\mathbb{T}}$, $k = 1,2,\dots$, then there exists piecewise rd-continuous input control $\widehat{U}(.)$ such that

$$z_{\alpha_{k+1}} = \emptyset(t_{k+1}, s_k)\left(\emptyset(s_k, t_{k+1})(z_{\alpha_k} + z_{\alpha_{k+1}})\right) + \int_{s_k}^{t_{k+1}} \emptyset(t_{k+1},\sigma(\tau))Q(\tau)\ \widehat{U}(\tau)\Delta\tau,$$

Which gives $z_{\alpha_k}^* z_{\alpha_k} = 0, k = 1,2,\dots,m$ Hence it is contradicts $z_{\alpha_k} \neq 0, k = 1,2,\dots,m,$ and therefore, the matrices $\mathcal{W}_k(s_k, t_{k+1})$ are invertible.

3.2 Controllability of non-linear impulsive system

Definition 3.4 A function $z(t) \in PC(I, \mathbb{R}^{n^2})$ is known as the solution of the system (2.1) is z(t) satisfies $z(t_0) = z_0, z(t_k^+) = [I_n \otimes R_k]z(t_k)$, t=$t_k$, k = 1, 2, …, m ; and z(t) is the solution of

$$z(t) = \emptyset(t,0)z_0 + \int_0^t \emptyset(t,\sigma(\tau))\left(H(\tau, x(\tau) + Q(\tau)\widehat{U}(\tau)\right)\Delta\tau, \forall\, t \in [0, t_0]_{\mathbb{T}}. \qquad (3.11)$$

$$z(t) = \emptyset(t, s_k)[I_n \otimes R_k]z(t_k) + \int_{s_k}^{t} \emptyset(t,\sigma(\tau))\left(H(\tau, x(\tau) + Q(\tau)\widehat{U}(\tau)\right)\Delta\tau,$$
$$\forall\, t \in (s_k, t_{k+1}]_{\mathbb{T}}, \qquad k = 1,2,\dots,m. \qquad (3.12)$$

we need the following conditions.

(C1): The non -linear function $H: J_1 \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, $J_1 = \cup_{k=0}^{m} (s_k, t_{k+1}]_{\mathbb{T}}$ is continuous and ∃ a ⨁ve constant such that

$\|H(t,z) - H(t,y)\| \le M_H\|z - y\|, \qquad \forall z, y \in \mathbb{R}^{n^2}, t \in J_1.$

Also, ∃ a ⨁ve constant L_H such that $\|H(t,z)\| \le L_H$, $\forall\, t \in J_1$ and $z \in \mathbb{R}^{n^2}$.

(C2): The function $g_k: I_k \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ are continuous and ∃ a ⨁ve constant such that $L_{g_k}, k = 1,2,\dots$; such that

$$\left\|\left([I_n \otimes R_k]z(t_k)\right) - \left(([I_n \otimes R_k]y(t_k))\right)\right\| \le M_{g_k}\|z - y\|,$$
$$\forall z, y \in \mathbb{R}^{n^2}, t \in I_k, k = 1,2,\dots;$$

Also, ∃ a ⨁ve constant L_g such that $\left\|\left([I_n \otimes R_k]z(t_k)\right)\right\| \le L_g$, $\forall\, t \in I_k$and $z \in \mathbb{R}^{n^2}$.

(C3): The linear system is controllable on $[0, T]$. Also, ∃ a positive constant δ_0, δ_1, such that $\|\mathcal{W}_0^{-1}(0, t_1)\| \le \delta_0$, $\|\mathcal{W}_k^{-1}(s_k, t_{k+1})\| \le \delta_1$.

Lemma 3.3 If all the condition (C1) -(C3) are satisfied, then the control input

$$\widehat{U}(t) = Q^*(\tau)\emptyset^*(t_1,\sigma(\tau))\mathcal{W}_0^{-1}(0,t_1)\left[z_{t_1} - \emptyset(t_1, 0)z_0 - \int_0^{t_1} \emptyset(t_1,\sigma(\tau)H(\tau, z(\tau))\Delta\tau\right],$$
$$\forall\, t \in [0, t_0]_{\mathbb{T}} \qquad (3.13)$$

Transfer the system (3.11) from z_0 *to* z_{t_1}at the time t_1. and, also, the estimate for the control input $\widehat{U}(t)$ is $M_{\widehat{U}}^0$, where

$M_{\widehat{U}}^0 = L_Q M \delta_0 (\|z_{t_1}\| + L\|z_0\| + LL_H T)$.

Proof. The proof of this lemma is the significance of lemma 3.1. Hence, we misplaced.

Lemma 3.4 If all the condition (C1) -(C3) are satisfied, then the control input

$$\widehat{U}(t) = Q^*(\tau)\emptyset^*\left(t_{k+1},\sigma(\tau)\right)\mathcal{W}_k^{-1}(s_k,t_{k+1})\Bigg[z_{t_{k+1}} - \emptyset(t_{k+1},s_k)[I_n \otimes R_k]z(t_k) - \int_{s_k}^{t_{k+1}} \emptyset(t_{k+1},\sigma(\tau)H(\tau,z(\tau))\Delta\tau\Bigg],$$
$$\forall\, t \in (s_k,t_{k+1}]_{\mathbb{T}}, k = 1,2,\dots,m, \qquad (3.14)$$

Transfer the system (3.8) from z_0 to z_{t_1}at the time t_{k+1}, where $t_{m+1} = T$ Also, the estimate for the control input $\widehat{U}(t)$ is $M_{\widehat{U}}^1$, where

$$M_{\widehat{U}}^1 = \max_{1\le k\le m}\left(L_Q L\delta_1\left(\|z_{t_{k+1}}\| + LL_g + LL_H T\right)\right).$$

Proof. The proof of this lemma is the significance of lemma 3.2. Hence, we misplaced.

Theorem 3.1 If all the condition (C1) -(C3) are satisfied, then the control system (2.1) is controllable provided

$$\max_{0\le j\le 1}\left(\frac{L^3L_Q^2\delta_j M_H T e_\Omega(T,0)}{\Omega}\right) < 1.$$

Proof. Let Consider a subset $\mathcal{B}' \subseteq PC$ such that

$$\mathcal{B}' = \left\{z \in PC\left(I,\mathbb{R}^{n^2}\right): \|z\|_{PC} \le \beta'\right\}.$$

Where

$$\beta' = \max\left(LL_g + \left(LL_H + LL_Q L_{\widehat{U}}^1\right)T, L\|z_0\| + \left(LL_H + LL_Q L_{\widehat{U}}^0\right)T, L_g\right).$$

Now, we define the operator $\mathcal{G}':\mathcal{B}' \to \mathcal{B}'$given by

$$(\mathcal{G}'z)(t) = \emptyset(t,0)z_0 + \int_0^t \emptyset\left(t,\sigma(\tau)\right)\left(H\left(\tau,z(\tau)\right) + Q(\tau)\widehat{U}(\tau)\right)\Delta\tau,\ \ \forall\, t \in [0,t_0]_{\mathbb{T}}. \quad (3.15)$$

$$(\mathcal{G}'z)(t) = g_k\left(t, \emptyset(t_k,s_{k-1})[I_n \otimes R_k]z(t_{k-1}^-) + \int_{s_{k-1}}^{t_k} \emptyset\left(t_k,\sigma(\tau)\right)\left(H\left(\tau,z(\tau)\right) + Q(\tau)\widehat{U}(\tau)\right)\Delta\tau,\right)$$
$$\forall\, t \in (s_k,t_k]_{\mathbb{T}}, k = 1,2,\dots,m \qquad (3.16)$$

$$(\mathcal{G}'z)(t) = \emptyset(t,s_k)[I_n \otimes R_k]z(t_k) + \int_{s_k}^t \emptyset\left(t,\sigma(\tau)\right)\left(H\left(\tau,z(\tau)\right) + Q(\tau)\widehat{U}(\tau)\right)\Delta\tau,$$
$$\forall\, t \in (s_k,t_{k+1}]_{\mathbb{T}}, k = 1,2,\dots,m \qquad (3.17)$$

We want to show that $\mathcal{G}':\mathcal{B}' \to \mathcal{B}'$.Now, for $t \in (s_k,t_{k+1}]_{\mathbb{T}}, k = 1,2,\dots$, and $z \in \mathcal{B}'$, we get

$$\|(\mathcal{G}'z)(t)\| \le \|\emptyset(t,s_k)\|\|[I_n \otimes R_k]z(t_k)\| + \int_{s_k}^t \|\emptyset\left(t,\sigma(\tau)\right)\|\|H\left(\tau,z(\tau)\right)\|\Delta\tau + \int_{s_k}^t \|\emptyset\left(t,\sigma(\tau)\right)\|\|Q(\tau)\|\|\widehat{U}(\tau)\|\Delta\tau$$
$$\le LL_g + LL_H(t-s_k) + LL_Q L_{\widehat{U}}^1(t-s_k).$$

Hence,

$$\|(\mathcal{G}'z)(t)\|_{PC} \le\ LL_g + \left(LL_H + LL_Q L_{\widehat{U}}^1\right)T. \qquad (3.18)$$

Now, for $t \in [0,t_0]_{\mathbb{T}}$ and $z \in \mathcal{B}'$, we get

$$\|(\mathcal{G}'z)(t)\| \le \|\emptyset(t,0)\|\|z_0\| + \int_0^t \|\emptyset\left(t,\sigma(\tau)\right)\|\|H\left(\tau,z(\tau)\right)\|\Delta\tau + \int_0^t \|\emptyset\left(t,\sigma(\tau)\right)\|\|Q(\tau)\|\|\widehat{U}(\tau)\|\Delta\tau$$
$$\le L\|z_0\| + LL_H t + LL_Q L_{\widehat{U}}^1 t.$$

Hence,

$$\|(\mathcal{G}'z)(t)\|_{PC} \le\ L\|z_0\| + \left(LL_H + LL_Q L_{\widehat{U}}^1\right)T. \qquad (3.19)$$

Similarly, for $t \in (s_k,t_{k+1}]_{\mathbb{T}}$ and $z \in \mathcal{B}'$, we get

$$\|(\mathcal{G}'z)(t)\| \leq L_g. \tag{3.20}$$

After brief the above inequalities (3.18) -(3.20), we have

$\|(\mathcal{G}'z)(t)\| \leq \beta'$.

Therefore, $\mathcal{G}':\mathcal{B}' \to \mathcal{B}'$. for $z, x \in \mathcal{B}', t \in (s_k, t_{k+1}]_{\mathbb{T}}, k = 1,2, \dots, m,$ we get

$$\begin{aligned}
&\|(\mathcal{G}'z)(t) - (\mathcal{G}'x)(t)\| \\
&\quad \leq \|\emptyset(t,s_k)\|\|[I_n \otimes R_k]z(t_k) - [I_n \otimes R_k]x(t_k)\| \\
&\quad + \int_{s_k}^{t} \|\emptyset(t,\sigma(\tau))\|\|H(\tau,z(\tau)) - H(\tau,x(\tau))\|\Delta\tau \\
&\quad + \int_{s_k}^{t} \|\emptyset(t,\sigma(\tau))\|\|Q(\tau)\|\|Q^*(\tau)\|\|\emptyset^*(t_{k+1},\sigma(\tau))\|\|\mathcal{W}_k^{-1}(s_k,t_{k+1})\| \\
&\quad \times \Big[\|\emptyset(t_{k+1},s_k)\|\|[I_n \otimes R_k]z(t_k) - [I_n \otimes R_k]x(t_k)\| \\
&\quad + \int_{s_k}^{t_{k+1}} \|\emptyset(t_{k+1},\sigma(s)\|\|H(s,z(s) - H(s,x(s)\|\Delta s\Big]\Delta\tau \\
&\leq LM_{g_k}\|z(t_k^-) - x(t_k^-)\| + LM_H\int_{s_k}^{t}\|z(\tau) - x(\tau)\|\Delta\tau \\
&\quad + L^3L_Q^2\delta_1\int_{s_k}^{t}\Big[M_{g_k}\|z(t_k^-) - x(t_k^-)\| + M_H\int_{s_k}^{t_{k+1}}\|z(s) - x(s)\|\Delta s\Big]\Delta\tau \\
&\leq LM_{g_k}e_\Omega(t_k,s_k)\|z-x\|_{PC} + \frac{LM_He_\Omega(t,s_k)\|z-x\|_{PC}}{\Omega} + L^3L_Q^2\delta_1(t-s_k) \\
&\times\Big[M_{g_k}e_\Omega(t_k,s_k)\|z-x\|_{PC} + \frac{L_H\|z-x\|_{PC}}{\Omega}(e_\Omega(t_{k+1},s_k) - 1)\Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(\mathcal{G}'z)(t) - (\mathcal{G}'x)(t)\|_{PC} \\
&\quad \leq \left[\frac{LM_{g_k}}{e_\Omega(s_k,t_k)}(1 + L^2L_Q^2\delta_1T) + \frac{LM_H}{\Omega}(1 + L^2L_Q^2\delta_1T(e_\Omega(T,0)))\right]\|z-x\|_{PC}
\end{aligned} \tag{3.21}$$

for $z, x \in \mathcal{B}', t \in [0,t_1]_{\mathbb{T}},$ we get

$$\begin{aligned}
&\|(\mathcal{G}'z)(t) - (\mathcal{G}'x)(t)\| \\
&\quad \leq \int_0^t \|\emptyset(t,\sigma(\tau))\|\|H(\tau,z(\tau)) - H(\tau,x(\tau))\|\Delta\tau \\
&\quad + \int_0^t \|\emptyset(t,\sigma(\tau))\|\|Q(\tau)\|\|Q^*(\tau)\|\|\emptyset^*(t_1,\sigma(\tau))\|\|\mathcal{W}_0^{-1}(0,t_1)\| \\
&\quad \times\left[\int_0^{t_1}\|\emptyset(t_{k+1},\sigma(s)\|\|H(s,z(s) - H(s,x(s)\|\Delta s\right]\Delta\tau \\
&\qquad \leq LM_H\|z-x\|_{PC}\int_0^t e_\Omega(\tau,0)\Delta\tau + L^3L_Q^2\delta_0M_Ht_1\|z-x\|_{PC}\int_0^{t_1}e_\Omega(s,0)\Delta s \\
&\quad \leq \frac{LM_He_\Omega(t,0)\|z-x\|_{PC}}{\Omega} + \frac{L^3L_Q^2\delta_0M_Ht_1e_\Omega(t_1,0)\|z-x\|_{PC}}{\Omega},
\end{aligned}$$

Which immediately gives

$$\|(\mathcal{G}'z)(t) - (\mathcal{G}'x)(t)\|_{PC} \leq \left[\frac{LM_H}{\Omega} + \frac{L^3L_Q^2\delta_0M_HT(e_\Omega(T,0))}{\Omega}\right]\|z-x\|_{PC}. \tag{3.22}$$

Correspondingly, for $t \in (s_k, t_{k+1}]_{\mathbb{T}}, k = 1,2, \dots, m,$ we have

$$
\begin{aligned}
&\|(\mathcal{G}'z)(t)-(\mathcal{G}'x)(t)\| \\
&\qquad \le M_{g_k}\Bigg(\|\emptyset(t_k,s_{k-1})\|\|[I_n\otimes R_{k-1}]z(t_{k-1})-[I_n\otimes R_{k-1}]x(t_{k-1})\| \\
&\qquad + \int_{s_{k-1}}^{t_k}\|\emptyset(t_k,\sigma(\tau))\|\|H(\tau,z(\tau))-H(\tau,x(\tau))\|\Delta\tau \\
&\qquad + \int_{s_{k-1}}^{t_k}\|\emptyset(t_k,\sigma(\tau))\|\|Q(\tau)\|\|Q^*(\tau)\|\|\emptyset^*(t_k,\sigma(\tau))\|\|\mathcal{W}_k^{-1}(s_{k-1},t_k)\| \\
&\qquad \times\Bigg[\|\emptyset(t_k,s_{k-1})\|\|[I_n\otimes R_k]z(t_{k-1}^{-1})-[I_n\otimes R_k]x(t_{k-1}^{-1})\| \\
&\qquad + \int_{s_{k-1}}^{t_k}\|\emptyset(t_{k+1},\sigma(s)\|\|H(s,z(s)-H(s,x(s)\|\Delta s\Bigg]\Delta\tau\Bigg) \\
&\le (1+L^2L_Q^2\delta_1T)LM_{g_k}\left(M_{g_k}e_\Omega(t_{k-1},t_k)+M_H\int_{s_{k-1}}^{t_k}e_\Omega(\tau,t_k)\Delta\tau\right)\|z-x\|_{PC}. \\
&\le (1+L^2L_Q^2\delta_1T)LM_{g_k}\left(M_{g_k}e_\Omega(t_{k-1},t_k)+\frac{M_H(1-e_\Omega(s_{k-1},t_k))}{\Omega}\right)\|z-x\|_{PC}
\end{aligned}
$$

Therefore,

$$
\|(\mathcal{G}'z)(t)-(\mathcal{G}'y)(t)\|_{PC}\le(1+L^2L_B^2\delta_1T)\left[\frac{LM_{g_k}^2}{e_\Omega(t_k,t_{k-1})}+\frac{LM_HM_{g_k}}{\Omega}\right]\|z-x\|_{PC}.\ (3.23)
$$

After Correspondingly the inequalities (3.21) -(3.23), we get

$$
\|(\mathcal{G}'z)(t)-(\mathcal{G}'y)(t)\|_{PC}\le M_{\mathcal{G}}'\|z-x\|_{PC}.
$$

Where

$$
M_{\mathcal{G}}'=\max_{0\le j\le 1}\left(\frac{L^3L_Q^2\delta_jM_HTe_\Omega(T,0)}{\Omega}\right).
$$

Therefore, $\mathcal{G}'$is a strict contraction mapping for sufficiently large Ω. Hence, the system (2.1) has a uniquely solution by Banach fixed points theorem.

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