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Application of Splines for Approximation

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Abstract: We consider the application of splines to function and integral function approximations, where the integral function refers to the indefinite integral whose lower limit is a constant and upper limit is a variable or function of a variable. Integral function approximation are very useful, one simple and immediate application refers to the construction of a table of integral for example the table of logarithms, normal distribution etc, other applications are possible in the evaluation of multiple integrals over triangular and tetrahedral regions to name a few.

Keywords: Approximation, Integral function, spline.

1. INTRODUCTION:

A qauartic spline interpolating first derivatives at the knots and the second derivatives between them is discussed. A feasible algorithm for the determination of the quartic spline for the second derivative parameter $\lambda = 0, 1/2, 1$ is presented. We consider the application of quartic splines to function and integral function approximations, where the integral function refers to the indefinite integral whose lower limit is a constant and upper limit is a variable or function of a variable. Integral function approximation are very useful in the construction of a table of integral, for example the table of logarithms, normal distribution etc, and other applications are possible in the evaluation of multiple integrals over triangular and tetrahedral regions. In this paper, we limit ourselves to simple application that is when the upper limit is a variable. A method based on spline integration to approximate the integral function is presented and compared the performance of quartic spline with standard cubic splines with the aid of graphs by using MATLAB programming.

SPLINES & QUARTIC SPLINE INTERPOLANT

A smooth curve generated using designated set of points that is flexible is spline. In computer graphics it is a composite curve with polynomial sections satisfying continuity conditions at the boundary. It is a combination of polynomial pieces with smoothness conditions. Spline of degree 1 has sections of linear polynomials linked to achieve continuity, as in figure below. The points $t_0, t_1, t_2, \ldots\ldots, t_n$ are called knots. In the following figure the function changes behaviour at eight points (knots).

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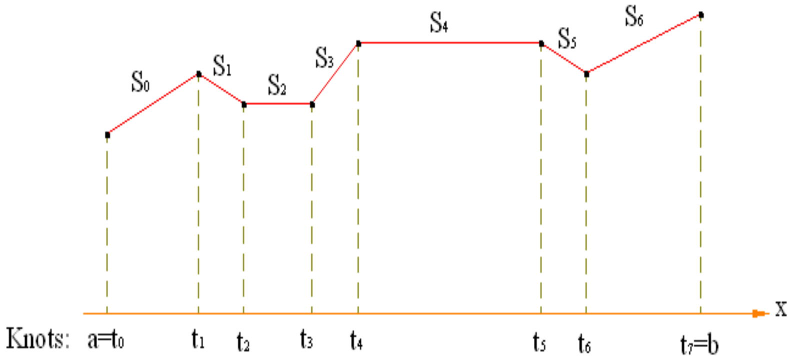
Figure First-degree spline function



The function is represented as

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \\ \vdots & \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

where $S_i(x) = a_i x + b_i$ because each piece of $S(x)$ is a linear polynomial, is piecewise linear.

Spline of degree k

S is termed a spline of degree k if: the domain of S is [a, b], $S, S', S'', \ldots., S^{(k-1)}$ are all continuous on [a, b], S is a polynomial of degree $\leq k$ on $[\, t_i, t_{i+1}\,]$ such that $a = t_0 < t_1 < t_2 < \ldots \ldots, < t_n = b$.

Higher degree splines will be helpful for more smoothness of approximation. For continuous mth derivative of approximating function, greater than or equal to $m + 1$th degree spline is to be chosen. Consider knots $t_0 < t_1 < t_2 < \ldots \ldots, < t_n$. A spline S with m continuous derivatives is to be obtained joining piecewise polynomials. At an interior knot t, To the left of t, $S(x) = p(x)$; to the right of t, $S(x) = q(x)$, where p, q are degree m polynomials. S^m is continuous leads to continuity of $S^{m-1}, S^{m-2}, \ldots . S', S$. Therefor, at t,

$$\lim_{x \to t^-} S^{(k)}(x) = \lim_{x \to t^+} S^{(k)}(x) \qquad (0 \leq k \leq m)$$

From which we conclude that

$$\lim_{x \to t^-} p^{(k)}(x) = \lim_{x \to t^+} q^{(k)}(x) \qquad (0 \leq k \leq m)$$

Since p and q are polynomials their derivatives of all orders are continuous and so the above equation is same as $p^{(k)}(t) = q^{(k)}(t) \qquad (0 \leq k \leq m)$

Hence by Taylor's theorem p, q to be the same polynomial

$$p(x) = \sum_{k=0}^{m} \frac{1}{k!} p^{(k)}(t)(x-t)^k = \sum_{k=0}^{m} \frac{1}{k!} q^{(k)}(t)(x-t)^k = q(x)$$

The same is applied at each interior knot $t_0, t_1, t_2, \ldots \ldots t_{n-1}, t_n$ that leads that S is merely the same polynomial all over the complete interval. Thus a piecewise polynomial of degree $m + 1$ with at most m continuous derivatives is necessary to have a spline function that is not just a single polynomial all over the interval. The ordinary polynomials do not assist well in the curve fitting.

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QUARTIC SPLINE INTERPOLANT

Consider f over [a,b] and let knots $\{x_i\}_{i=0}^{N+1}$ $a = x_0 < x_1 < x_2 < ..... < x_N < x_{N+1} = b$ be the (N+1) distinct points. Note that x_i's divide [a, b] into N+1 subintervals. A function S is said to be a quartic spline on the interval [a, b], if S, S', S'' and S''' are continuous in [a, b] and S is a polynomial of degree ≤ 4 in each knot interval $[x_{i-1}, x_i]$.

We now consider a standard form of quartic spline interpolant.

Given $f_i'(i = 0,1,2, \ldots .N + 1)$, $f''_{i+\lambda}(i = 0,1,2, \ldots .N), f_0$ and f_{N+1} , then there exists a unique quartic spline function S (x) such that

On each subinterval $[x_i, x_{i+1}], i = 0, 1, 2, \ldots, N$, S($x$) coincides with the quartic polynomial $S(x) = S_i(u) = a_i + b_i u + c_i u^2 + d_i u^3 + e_i u^4, u = \frac{x-x_i}{h}, h = x_{j+1} - x_j, 0 \leq u \leq 1.$

S, S' and S''satisfy the interpolatory conditions $S'(x_i) = S_i' = f_i', i = 0.1.2 \ldots \ldots N + 1$

$s''(x_i + \lambda h) = s''_{i+\lambda} = f''_{i+\lambda},\ i = 0, 1, 2 \ldots .N$ $s(x_0) = s_0 = f_0$, $s(x_{N+1}) = s_{N+1} = f_{N+1}$

when ever $0 \leq \lambda \leq 1, \lambda \neq \frac{3\pm\sqrt{3}}{6}$ and N is even if $\lambda = \frac{1}{2}$.

APPLICATIONS

Function Approximations

Let us consider the integral representation $\int_a^x f'(t)dt = f(x) - f(a),$ if $f(t) = \int f'(t)dt$

Clearly f(a) is arbitrary and if f(a) = 0, we can write $f(x) = \int_a^x f'(t)dt$ $with$ $f(t) = \int f'(t)dt$

Example 1: Let $f'(t) = \frac{1}{t}$ then $\int_1^x \frac{dt}{t} = f(x)$ since $f(t) = \int f'(t)dt = \int \frac{dt}{t}$ and $f(1) = 0$

Example2: Let $f'(t) = \frac{1}{t+1}$ then $\int_1^x \frac{dt}{t+1} = f(x) - f(1)$ where $f(t) = \int f'(t)dt = \int \frac{dt}{t+1} = \log(t+1)$ and $f(1) = log2$.

The above subtle points must be noted while applying the quartic spline algorithm to numerical integration. Thus quartic spline approximation for $F(x) = f(x) - f(a)$, with $f(t) = \int f'(t)dt$ is suggested for the application. Suitable caution must be taken while programming.

Now consider the Runge Function as an example $f(x) = \frac{1}{1+25\,x^2}, x\epsilon[-1,1]$

and find approximation of this by quartic spline.

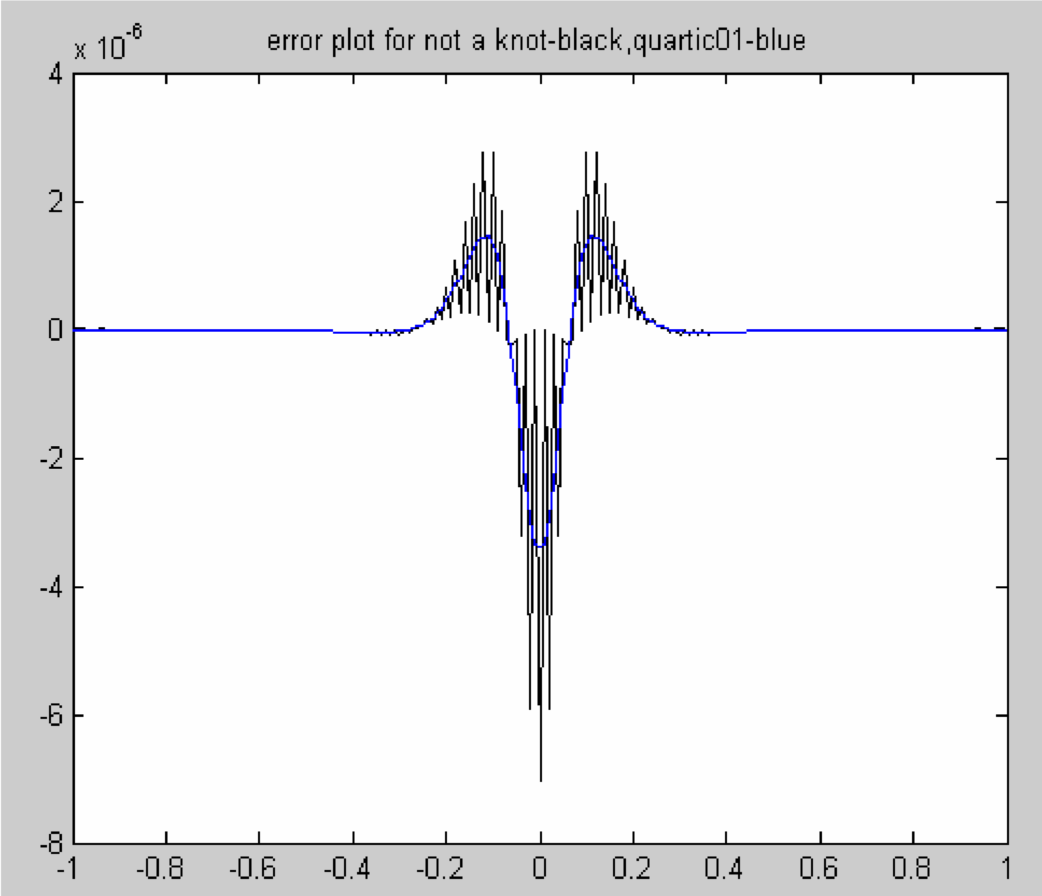
Let $F(x) = \int_{-1}^x f'(t)dt = \frac{1}{1+25\,x^2} - \frac{1}{26} = f(x) - \frac{1}{26}, x\epsilon[-1,1]$

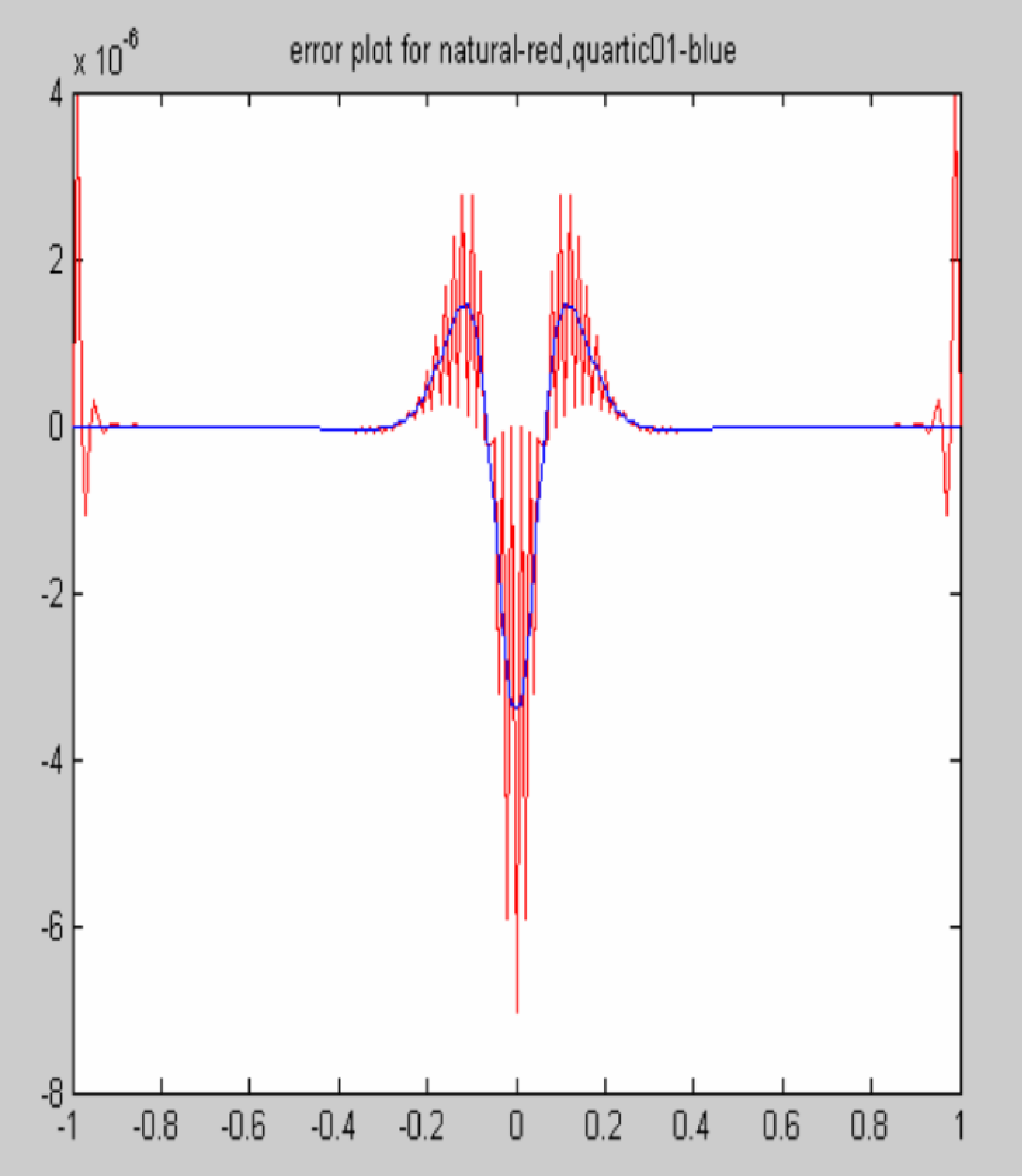
we observe that $F(-1) = F(1) = 0$ and $F'(x) = f'(x),$ $F''(x) = f''(x)$

Hence, the application of the quartic spline theory proposed in the previous section computes the sequence $\{F(x_i)\}_{i=1}^N$, assuming F(x_0)and F(x_{N+2}) as known values. Thus to obtain the approximations to Runge's function $\{f(x_i)\}_{i=0}^{N+2}$, we have to use the functional relation $f(x) = F(x) + \frac{1}{26}$. We have discussed this for the cases λ=0, 0.5, 1 through MATLAB.

ERROR PLOTS FOR CUBIC AND QUARTIC SPLINES

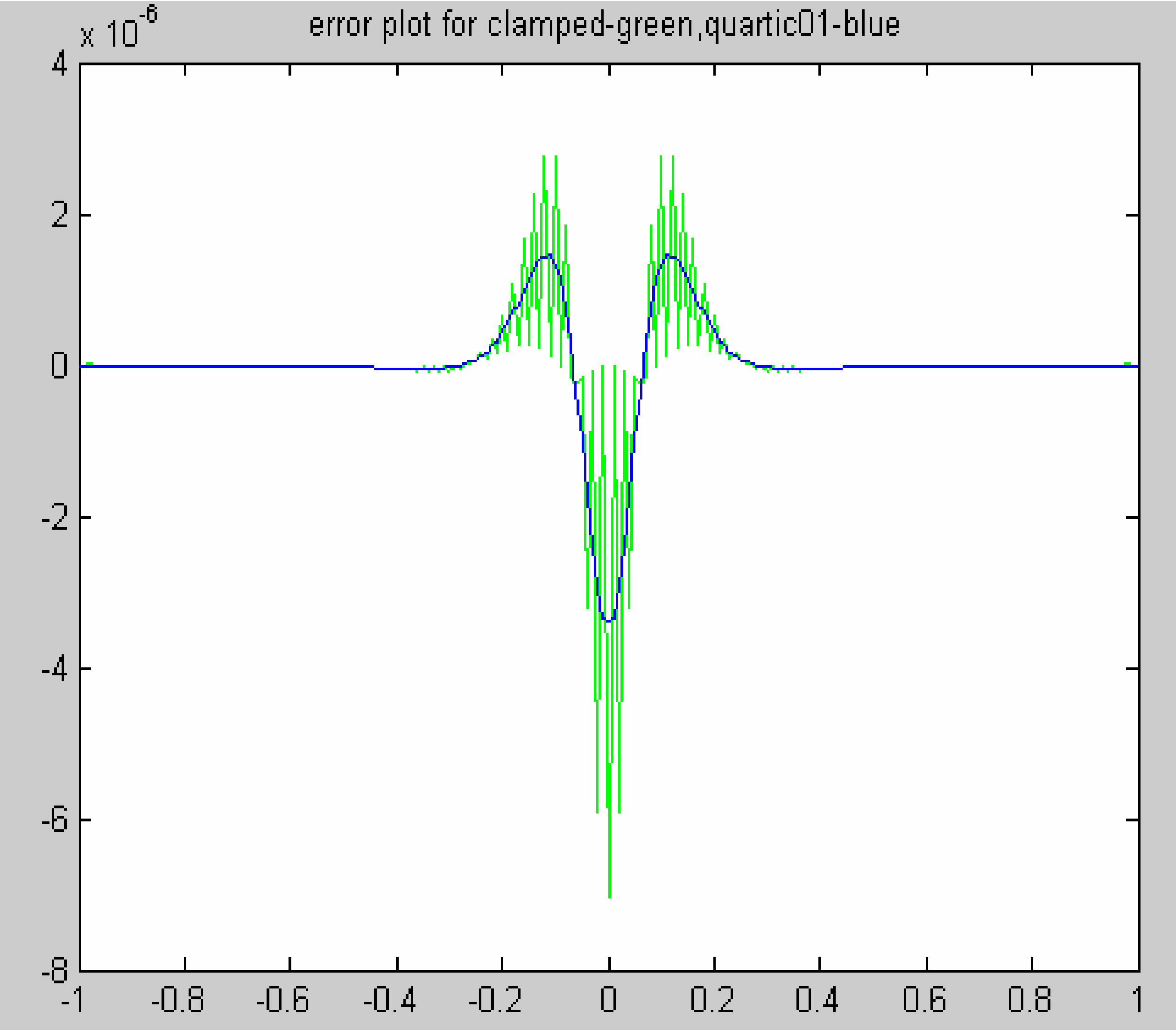
We have also displayed error plots for standard cubic splines (natural clamped and not a knot splines) and superposed these plots by quartic splines for λ=0, 1/2 and 1. We find that the performance of quartic splines is superior compared with the standard cubic splines.


error plot for not a knot-black,quartic01-blue
x 10^-6
4
2
0
-2
-4
-6
-8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



2. CONCLUSIONS

We have discussed the function approximation, integral function approximation using quartic spline interpolant. Performance of the proposed quartic spline is compared with the standard cubic splines (natural, clamped and not a knot splines).



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