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Further results on Odd Super Graceful Labeling of Graphs

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***Abstract:* A graph G with p vertices and q edges is said to have an odd super graceful labeling if there exists an injective function $f: V(G) \cup E(G) \to \{1, 2, 3, \dots, p+q\}$ such that $f(uv) = |f(u) - f(v)|$, for all $uv \in E(G)$ and all its vertex labels are odd. A graph that admits an odd super graceful labeling is called an odd super graceful graph. In this paper, we investigate the odd super gracefulness of some graphs obtained by graph operations such as identification, carona, vertex duplication and edge duplication.**

***Keywords.* Odd super graceful labeling, Odd super graceful graph.**

2010 Mathematics Subject Classification Number: 05C78.

1. INTRODUCTION

By a graph we mean a finite undirected graph $G(V(G), E(G))$ without loops or multiple edges with p vertices and q edges.

A path on n vertices is denoted by P_n. The complete bipartite graph $K_{1,n}$ is a star graph. $G \odot K_1$ is a graph obtained from the graph G by attaching a pendant vertex to each vertex of G. The graph $T_p^{(n)}$ is a tree formed from n copies of path on p vertices by joining an edge uu' between every pair of consecutive paths where u is a vertex in i^{th} copy of the path and u' is the corresponding vertex in the $(i+1)^{th}$ copy of the path. The H-graph H_n is a graph obtained from two paths $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ by adding an edge between $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ while n is odd and $u_{\frac{n}{2}+1}$ and $v_{\frac{n}{2}}$ while n is even. The graph $P_m@K_{1,n}$ is a graph obtained by identifying the pendant vertex of the path P_m with central vertex of $K_{1,n}$.

Duplicating a vertex v of a graph G produces a new graph G' by adding a new vertex v' and join v' to all the vertices of G adjacent to v. Duplicating of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$ [9].

The concept of graceful labeling has been introduced by Rosa [7] in 1967. A function f is a graceful labeling of a graph G with p vertices and q edges if f is an injection from the vertices of G to the set $\{0,1,2, \dots, q\}$ such that when each edge uv is assigned the label $|f(u) -$

$f(v)|$, the resulting edge labels are distinct. M.A. Perumal et al. have introduced and discussed about the super graceful labeling of a graph [4, 5, 6]. A bijective function $f: V(G) \cup E(G) \to \{1,2,\dots,p+q\}$ such that $f(uv) = |f(u) - f(v)|$, for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling.

In [2], R.B. Gnanajothi introduced the concept of odd graceful graphs and she has proved many results on this newly defined concept. A graph G is said to admit odd graceful labeling if $f: V(G) \to \{0,1,2,\dots,2q-1\}$ is injective and the induced function $f^*: E(G) \to \{1,3,5,\dots,2q-1\}$ defined as $f^*(uv) = |f(u) - f(v)|$, for all $uv \in E(G)$, is bijective. A graph which admits odd graceful labeling is called an odd graceful graph. In [2], K.M. Kathiresan has discussed odd gracefulness of ladders and graphs obtained from them by subdividing each step exactly once. In [7], C. Sekar has proved that the splitting graph of path P_n and the splitting graph of even cycle C_n are odd graceful graphs.

Motivated by these works, we introduce a new type of labeling called odd super graceful labeling of graphs in [1] and studied for some standard graphs. A graph G is said to have an odd super graceful labeling if there exists an injective function $f: V(G) \cup E(G) \to \{1,2,3,\dots,p+q\}$ such that $f(uv) = |f(u) - f(v)|$, for all $uv \in E(G)$ and all its vertex labels are odd. A graph that admits an odd super graceful labeling is called an odd super graceful graph. In this paper, we investigate the odd super gracefulness of some graphs obtained by graph operations such as vertex identification, carona, vertex duplication and edge duplication.

Theorem 1.1 [1] *Any unicycle graph is not an odd super graceful graph.*

2. Main Results

Proposition 2.1 *Let G be a graph obtained from a path P_n on n vertices by attaching a path on i vertices in each i^{th} vertex of the path $P_n, 1 \le i \le n$. Then G is an odd super graceful graph.*

Proof. Let $u_{i,j}, 1 \le j \le i, 1 \le i \le n$, be the vertices of i^{th} path P_i in G. Then G has $\frac{n(n+1)}{2}$ vertices and $\frac{n^2+n-2}{2}$ edges. So $|V(G) \cup E(G)| = n^2 + n - 1$.
Define $f: V(G) \cup E(G) \to \{1,2,3,\dots,n^2+n-1\}$ as follows:

For $1 \le i \le n$ and $1 \le j \le i$,

$$f(u_{i,j}) = \begin{cases} \frac{i^2-1}{2} + j, & if\ i\ and\ j\ are\ odd \\ n^2 + n + 1 - \frac{(i-1)^2}{2} - j, & if\ i\ is\ odd\ and\ j\ is\ even \\ n^2 + n - \frac{i^2}{2} + j, & if\ i\ is\ even\ and\ j\ is\ odd \\ \frac{i^2+2i}{2} - j + 1, & if\ i\ and\ j\ are\ even. \end{cases}$$

For $1 \le i \le n$ and $1 \le j \le i-1$,

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$$f(u_{i,j}u_{i,j+1}) = \begin{cases} n^2+n-i^2+i-2j, & if\ \ i\ \ is\ \ odd \\ n^2+n-i^2-i+2j, & if\ \ i\ \ is\ \ even. \end{cases}$$

For $1 \le i \le n-1$,

$$f(u_{i,1}u_{i+1,1}) = n^2+n-i^2-i.$$

Then f is an odd super graceful labeling of G.

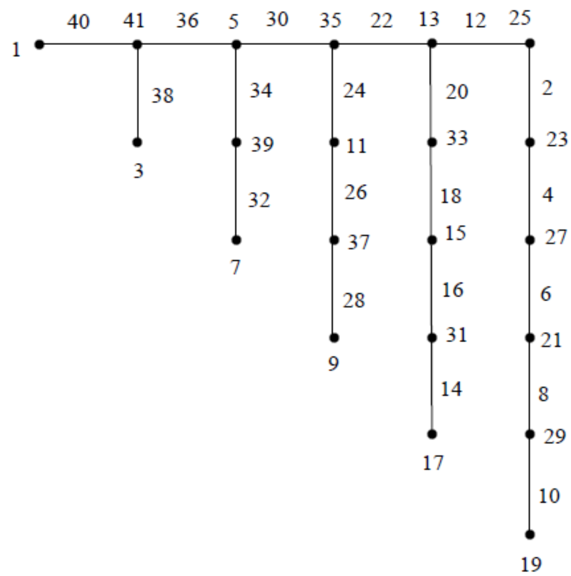

Figure 1 : An odd super graceful labeling of G when $n = 6$.

Proposition 2.2 *Let G be a graph obtained by identifying the central vertex of a copy of $K_{1,n}$ with each pendant vertex of $K_{1,3}$. Then G is an odd super graceful graph for any $n \ge 1$.*

Proof. Let $u_{i,1}, u_{i,2}, \ldots, u_{i,n}$ be the pendant vertices and x_i be the central vertex of the i^{th} copy of $K_{1,n}$, $1 \le i \le 3$. Let y be the central vertex of $K_{1,3}$. Then G has $3n+4$ vertices and $3n+3$ edges.

Define $f: V(G) \cup E(G) \to \{1,2,3, \ldots, 6n+7\}$ as follows:

$$f(u_{i,j}) = 2(n+1)(i-1)+2j+3, 1 \le i \le 3 and\ \ 1 \le j \le n,$$

$$f(x_i) = \begin{cases} 2n+5, & i=1 \\ 3, & i=2 \\ 1, & i=3, \end{cases}$$

$$f(y) = 4n+7,$$

$$f(x_iu_{i,j}) = \begin{cases} 2n+2-2j, & i=1\ \ and\ \ 1 \le\ \ j \le\ \ n \\ 2n+2+2j, & i=2\ \ and\ \ 1 \le\ \ j \le\ \ n \\ 4n+6+2j, & i=3\ \ and\ \ 1 \le\ \ j \le\ \ n \end{cases}$$

$$\text{and}\ \ f(yx_i) = \begin{cases} 2n+2, & i=1 \\ 4n+4, & i=2 \\ 4n+6, & i=3. \end{cases}$$

Then f is an odd super graceful labeling of G.

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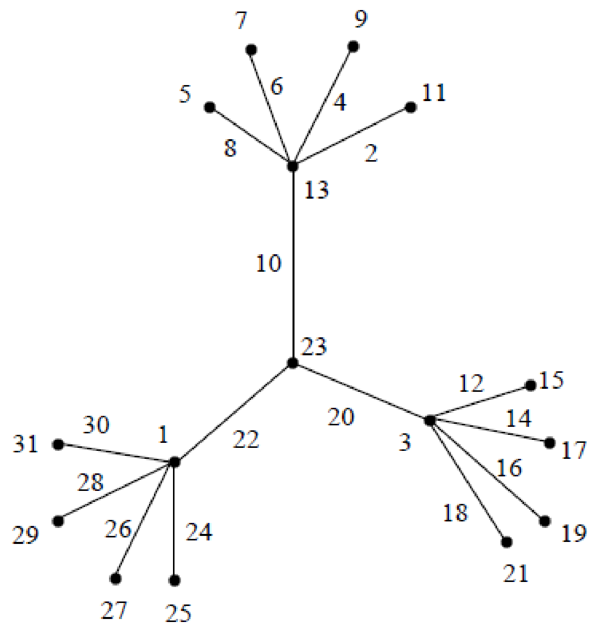

Figure 2: An odd super graceful labeling of G when $n = 4$.

Proposition 2.3 *For any* $n \geq 2, T_p(n)$ *is an odd super graceful graph.*

Proof. Let $u_{i,j}, 1 \leq i \leq n, 1 \leq j \leq p,$ be the vertices of the i^{th} copy of path on p vertices.
Define $f: V(T_p(n)) \cup E(T_p(n)) \to \{1,2,3, \ldots, 2np-1\}$ as follows:
For $1 \leq i \leq n$ and $1 \leq j \leq p,$

$$f(u_{i,j}) = \begin{cases} (i-1)p + j, & if\ i\ and\ j\ are\ odd \\ 2np - (i-1)p - j + 1, & if\ i\ is\ odd\ and\ j\ is\ even \\ 2np - ip + j, & if\ i\ is\ even\ and\ j\ is\ odd \\ 2(i-1)p - j + 1, & if\ i\ and\ j\ are\ even. \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq j \leq p-1,$

$$f(u_{i,j}u_{i,j+1}) = \begin{cases} 2np - 2(i-1)p - 2j, & if\ i\ is\ odd \\ 2np - (3i-2)p + 2j, & if\ i\ is\ even. \end{cases}$$

For $1 \leq i \leq n-1$ and any fixed $k, 1 \leq k \leq p,$

$$f(u_{i,k}u_{i+1,k}) = 2np - 2ip.$$

Then f is an odd super graceful labeling of $T_P(n)$.

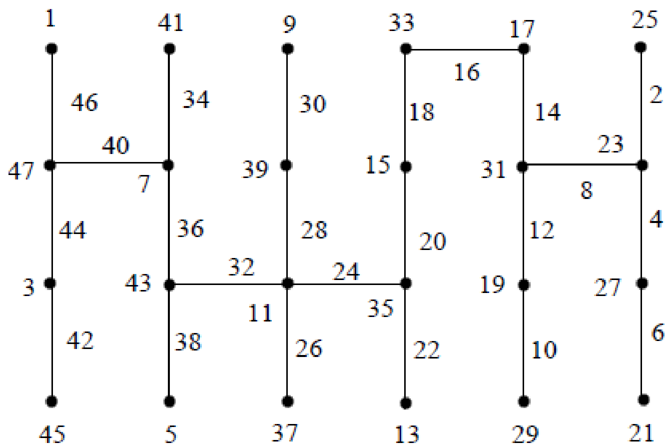

Figure 3: An odd super graceful labeling of $T_4(6)$.

Proposition 2.4 *For any $n \geq 2$, the H-graph H_n is an odd super graceful graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of length $n-1$ in H_n.
Case (i). n is odd.
Define $f: V(H_n) \cup E(H_n) \to \{1,2,3,\ldots,4n-1\}$ as follows:

$$f(u_i) = \begin{cases} i, & 1 \leq \ i \leq \ n \ and \ i \ is \ odd \\ 4n+1-i, & 1 \leq \ i \leq \ n \ and \ i \ is \ even, \end{cases}$$

$$f(v_i) = \begin{cases} 2n+i, & 1 \leq \ i \leq \ n \ and \ i \ is \ odd \\ 2n+1-i, & 1 \leq \ i \leq \ n \ and \ i \ is \ even, \end{cases}$$

$$f(u_i u_{i+1}) = 4n - 2i, 1 \leq i \leq n-1,$$

$$f(v_i v_{i+1}) = 2i, 1 \leq i \leq n-1$$

$$\text{and } f\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = 2n.$$

Thus f is an odd super graceful labeling of H_n.
Case (ii). n is even.
Define $f: V(H_n) \cup E(H_n) \to \{1,2,3,\ldots,4n-1\}$ as follows:

$$f(u_i) = \begin{cases} i, & 1 \leq \ i \leq \ n \ and \ i \ is \ odd \\ 4n+1-i, & 1 \leq \ i \leq \ n \ and \ i \ is \ even, \end{cases}$$

$$f(v_i) = \begin{cases} n+i, & 1 \leq \ i \leq \ n \ and \ i \ is \ odd \\ 3n+1-i, & 1 \leq \ i \leq \ n \ and \ i \ is \ even, \end{cases}$$

$$f(u_i u_{i+1}) = 4n - 2i, 1 \leq i \leq n-1,$$

$$f(v_i v_{i+1}) = 2n - 2i, 1 \leq i \leq n-1$$

$$\text{and } f\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = 2n.$$

Thus f is an odd super graceful labeling of H_n.

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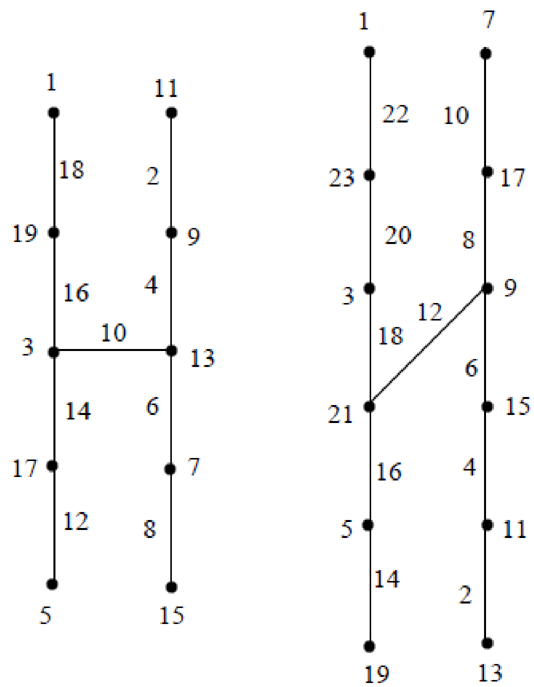

Figure 4: An odd super graceful labeling of H_5 and H_6.

Proposition 2.5 *For any* $m, n \geq 1, (P_m@K_{1,n}) \odot K_1$ *is an odd super graceful graph.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of P_m and $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. $P_m@K_{1,n}$ is obtained by identifying the vertex u_m with the central vertex of $K_{1,n}$. Let x_i and y_j be the pendant vertices attached at u_i and v_j respectively, $1 \leq i \leq m$ and $1 \leq j \leq n$ in $G = \left(P_m@K_{1,n}\right) \odot K_1$.
Define $f: V(G) \cup E(G) \rightarrow \{1,2,3, \ldots, 4m+4n-1\}$ as follows:

$$f(u_i) = \begin{cases} 2i-1, & 1 \leq i \leq m,\ i \text{ is odd and } m \text{ is even} \\ & \quad \text{or } i \text{ is even and } m \text{ is odd} \\ 4m+4n-2i+1, & 1 \leq i \leq m,\ i \text{ and } m \text{ are odd} \\ & \quad \text{or } i \text{ and } m \text{ are even,} \end{cases}$$

$$f(x_i) = \begin{cases} 4m+4n-2i+1, & 1 \leq i \leq m \text{ and } i \text{ is odd} \\ 2i-1, & 1 \leq i \leq m \text{ and } i \text{ is even,} \end{cases}$$

$f(v_i) = 2m + 4i - 3, 1 \leq i \leq n,$
$f(y_i) = 2m + 4n - 4i + 3, 1 \leq i \leq n,$
$f(u_i x_i) = 4m + 4n - 4i + 2, 1 \leq i \leq m,$
$f(u_i u_{i+1}) = 4m + 4n - 4i, 1 \leq i \leq m - 1,$
$f(u_m v_i) = 4n - 4i + 4, 1 \leq i \leq n$

$$\text{and } f(v_i y_i) = \begin{cases} 4n+6-8i, & 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ 4n-6-8(n-i), & \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

Then f is an odd super graceful labeling of G.

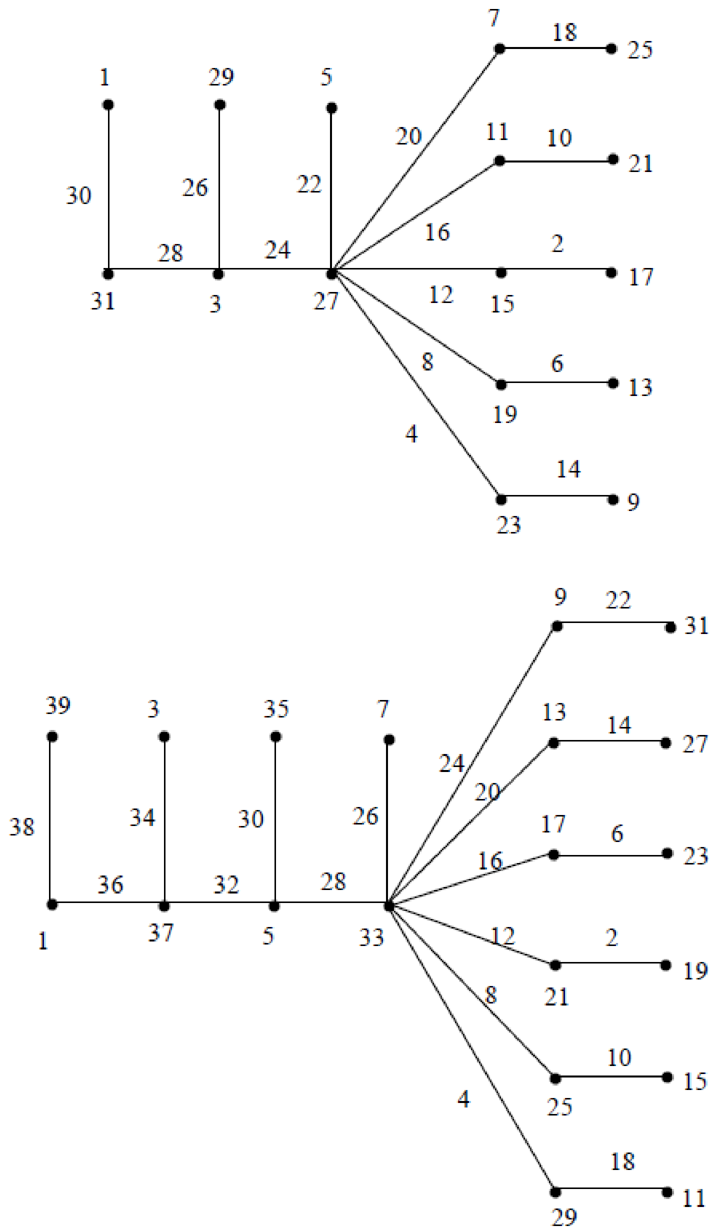

Figure 5: An odd super graceful labeling of $(P_3@K_{1,5}) \odot K_1$ and $(P_4@K_{1,6}) \odot K_1$.

Corollary 2.6 *$K_{1,n} \odot K_1$ is an odd super graceful graph.*

Proof. By taking $m = 1$ in Proposition 2.5, the result follows.

Proposition 2.7 *Let G be a graph obtained from a path P_n by identifying a pendant vertex of a copy of $K_{1,m}$ at each of its vertices. Then G is an odd super graceful graph for any integer $m \geq$*

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2 *and even integer* $n \geq 2$.

Proof. Let u_i be the central vertex and $v_{i,j}, 1 \leq j \leq m$ be the pendant vertices of the i^{th}copy of $K_{1,m}, 1 \leq i \leq n$ in which $v_{i,m}$ is identified with the i^{th} vertex of the path P_n in G.
Define $f: V(G) \cup E(G) \rightarrow \{1,2,3, \dots, 2(m+1)n - 1\}$ as follows:
For $1 \leq i \leq n$,

$$f(u_i) = \begin{cases} (m+1)(i-1)+1, & if \;\; i \;\; is \;\; odd \\ (m+1)(2n-i)+1, & if \;\; i \;\; is \;\; even. \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq j \leq m-1$,

$$f(v_{i,j}) = \begin{cases} (m+1)(2n-i+1)-2j+1, & if \;\; i \;\; is \;\; odd \\ (m+1)(i-2)+2j+3, & if \;\; i \;\; is \;\; even. \end{cases}$$

For $1 \leq i \leq n$,

$$f(v_{i,m}) = \begin{cases} (m+1)(2n-i-1)+3, & if \;\; i \;\; is \;\; odd \\ (m+1)(i-2)+3, & if \;\; i \;\; is \;\; even. \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq j \leq m-1$,

$$f(u_i v_{i,j}) = \begin{cases} 2(m+1)(n-i+1)-2j, & if \;\; i \;\; is \;\; odd \\ 2(m+1)(n-i+1)-2j-2, & if \;\; i \;\; is \;\; even. \end{cases}$$

For $1 \leq i \leq n$,

$$f(u_i v_{i,m}) = \begin{cases} 2(m+1)(n-i)+2, & if \;\; i \;\; is \;\; odd \\ 2(m+1)(n-i+1)-2, & if \;\; i \;\; is \;\; even. \end{cases}$$

For $1 \leq i \leq n-1$,

$$f(v_{i,m} v_{i+1,m}) = 2(m+1)(n-i).$$

Thus f is an odd super graceful labeling of G.

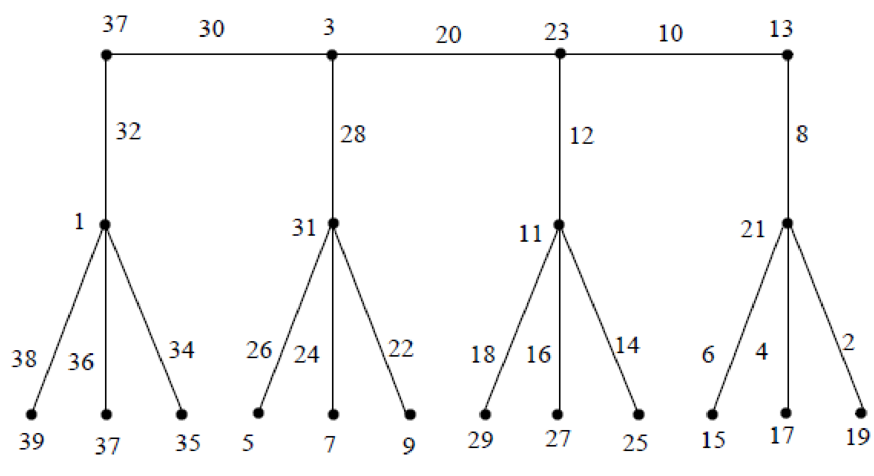

Figure 6: An odd super labeling of G when n = 4 and m = 3

Theorem 2.8 *Let* G *be a graph obtained by duplicating any vertex* v *of* P_n *by a vertex for* $n \geq 2$. *Then* G *is an odd super graceful graph only when* v *is a pendant vertex of* P_n.

Proof. If v is not a pendant vertex of P_n, then G is an unicyclic graph and by Theorem 1.1, G is not an odd super graceful graph.

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Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and $v_{n'}$ be the duplicating vertex of v_n in G.
Define $f: V(G) \cup E(G) \rightarrow \{1,2, \ldots ,2n+1\}$ as follows:

$$f(v_i) = \begin{cases} i, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2n - i + 3, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v'_n) = \begin{cases} n+2, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even,} \end{cases}$$

$$f(v_i v_{i+1}) = 2n - 2i + 2, 1 \leq i \leq n-1$$

$$\text{and } f(v_{n-1} v'_n) = 2.$$

Then f is an odd super graceful labeling of G.

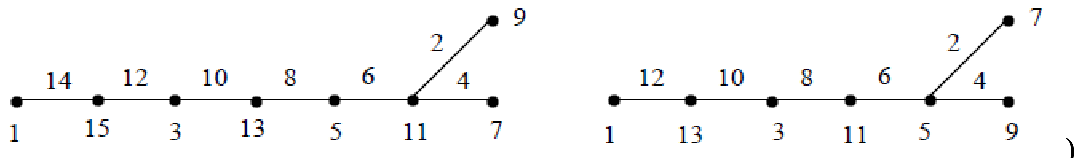

Figure 7: An odd super graceful labeling of G when $n = 7$ and 6.

Theorem 2.9 *Let G be any graph obtained by duplicating any edge e of P_n by an edge for $n \geq 3$. Then G is an odd super graceful graph only when e is a pendant edge of P_n.*

Proof. If e is not a pendant edge of P_n, then G is an unicyclic graph and hence by Theorem 1.1, G is not an odd super graceful graph.
Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and $v'_{n-1} v'_n$ be the duplicating edge of $v_{n-1} v_n$ in G.
Define $f: V(G) \cup E(G) \rightarrow \{1,2,3, \ldots ,2n+3\}$ as follows:

$$f(v_i) = \begin{cases} i, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2n - i + 5, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v'_{n-1}) = \begin{cases} n+2, & \text{if } n \text{ is odd} \\ n+3, & \text{if } n \text{ is even,} \end{cases}$$

$$f(v_i v_{i+1}) = 2n - 2i + 4, 1 \leq i \leq n-1,$$

$$f(v_{n-2} v'_{n-1}) = 4$$

$$\text{and } f(v'_{n-1} v'_n) = 2.$$

Thus f is an odd super graceful labeling of G.

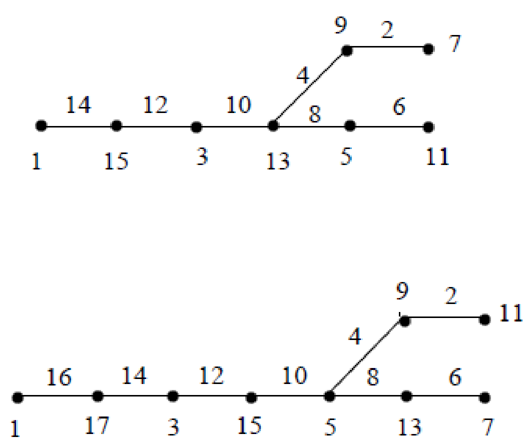

Figure 8: An odd super graceful labeling of G for $n = 6$ and 7.

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