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Cyclic Contractions And Fixed Point Theorems In Banach Spaces

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Abstract—In this manuscript we have proved the existence and uniqueness of some fixed point theorems for the cyclic operators defined in a closed subset of a Banach Space. Fixed point theorems for some contractions are introduced and given some examples.
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Keywords—Fixed Point, Contractive Mapping, Cylic Contraction, Banach Spaces

1. INTRODUCTION

In 2003 Kirk et.al introduced the notion of cyclic representation and characterized the Banach Contraction Principle in context of cyclic mapping.The theory of existence and uniqueness of fixed points has been developing since the work of Banach [9] in 1922 and numerous results have been obtained so far.Various types of cyclic contractions acting on complete metric spaces have been defined and studied thoroughly from this point of view[1]- [12].Now we extend our view to prove fixed point results for cyclic contraction in complete metric Spaces which generalize the results for cyclic contractions in Banach Space.

2. PRELIMINARIES

Definition 2.1[see (11)]
Let K be a subset of a Banach space X.An operator T defined on K is said to belong to the class D(p,q) if $\|Tx-Ty\| \le p\|x-y\|+q\{\|x-Tx\|+\|y-Ty\|\} \to$ (2.1) for all x and y in K,Where $0 \le p,q \le 1$ .If an operator T is in class D(k,0) with $0<k<1$, then T is contraction with 0<k<1.
Definition 2.2
Let K_1 and K_2 be closed subsets of a Banach space X.An operator T defined on K is said to belong to the class D(p,q,r) if $\|Tx-Ty\| \le p\|x-y\|+q\{\|x-Tx\|+\|y-Ty\|\}+r\{\|x-Ty\|+\|Tx-y\|\}$ -----------(2.2) for all x and y in K,Where $0 \le p,q,r \le 1, p+2q+2r \le 1 \ and \ q>0.$
Definition 2.3

Let K_1 and K_2 be closed subsets of a Banach Space X.An operator $T: K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ is said to belong to class D(p,q) if it satisfies

$$\|Tx - Ty\| \le p\|x - y\| + q\{\|x - Tx\| + \|y - Ty\|\} \text{ ----------------(2.3)}$$

for all $x \in K_1$ and $y \in K_2$ where $0 \le p, q \le 1$. It is clear that if T belongs to the class D(k,0) with $0 < k < 1$,then T is a cyclic contraction.

Example 2.4:

$$Let\ K_1 = [0, \frac{1}{2}]\ and\ K_2 = [\frac{1}{3}, 1].$$

Define the operator T as follows:

$$T(x) = \{\frac{2}{5}, if\ 0 \le x \le \frac{1}{2}; \quad \frac{2}{3}(1 - x), if\ \frac{1}{2} < x \le 1;\}$$

To Prove that T is in the class of $D(\frac{1}{4}, \frac{1}{4})$.

Take $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{3}, \frac{1}{2}]$.

Then $\left\|Tx - Ty\right\| = \left|\frac{2}{5} - \frac{2}{5}\right| = 0.$

Now $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$. Then $\|Tx - Ty\| = \left|\frac{2}{5} - \frac{2}{3} + \frac{2}{3}y\right| = \left|\frac{2}{3}y - \frac{4}{15}\right|$

$$= \left|\frac{1}{4}x - \frac{1}{4}y + \frac{1}{4}x - \frac{1}{10} + \frac{5}{12}y - \frac{1}{6}\right|$$

$$\le \frac{1}{4}|x - y| + \frac{1}{4}(\left|x - \frac{2}{5}\right| + \left|\frac{5}{3}y - \frac{2}{3}\right|)$$

$$= \frac{1}{4}|x - y| + \frac{1}{4}(|x - Tx| + |y - Ty|)$$

Which implies that T has a unique fixed point $P = \frac{2}{5}$.

Definition 2.5

Let K_1 and K_2 be closed subsets of a Banach space X.An operator $T: K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ is is said to belong to the class D(p,q,r) if it satisfies $\|Tx - Ty\| \le p\|x - y\| + q\{\|x - Tx\| + \|y - Ty\|\} + r\{\|x - Ty\| + \|Tx - y\|\}$ -----------(2.4) for all $x \in K_1$ and $y \in K_2$, where $0 \le p, q \le 1$. It is clear that if T belongs to the class D(k,0,0) with 0<k<1,then T is cyclic contraction.

Definition 2.6

Let K_1 and K_2 be closed subsets of a Banach space X.An operator T defined on K is said to belong to the class D(p,q,r,s) if

$$\|T_x - T_y\| \le p\|x - y\| + q\{\|x - T_x\| + \|y - T_y\|\} + r\{\|x - T_y\| + \|T_x - y\|\} + s\{\frac{\|x - T_y\| + \|y - T_x\|}{2}\} \text{ -----------}$$

(2.6) for all x and y in K,

Where $0 \le p,q,r,s \le 1, p+2q+2r+2(s/2) \le 1$ *and* $q > 0$.

Definition 2.7

Let K_1 and K_2 be closed subsets of a Banach space X.An operator $T: K_1 \cup K_2 \to K_1 \cup K_2$ *with* $T(K_1) \subset K_2$ *and* $T(K_2) \subset K_1$ is is said to belong to the class D(p,q,r,s) if it satisfies

$$\|T_x - T_y\| \le p\|x-y\| + q\{\|x-T_x\| + \|y-T_y\|\} + r\{\|x-T_y\| + \|T_x - y\|\} + s\{\frac{\|x-T_y\| + \|y-T_x\|}{2}\} \text{ -----------}$$

(2.4) for all , $x \in K_1$ *and* $y \in K_2$, *where* $0 \le p,q,r \le 1$. It is clear that if T belongs to the class D(k,0,0,0) with 0<k<1,then T is cyclic contraction.

3. MAIN RESULTS

Proposition 3.1

Let K_1 and K_2 be closed subset of a Banach space X.An operator $T: K_1 \cup K_2 \to K_1 \cup K_2$ *with* $T(K_1) \subset K_2$ *and* $T(K_2) \subset K_1$ and satisfies (2.4) with $0 \le p,q,r,s < 1$ *and* $p+2q+2r+2(s/2) < 1$. If $F(T) = \{x \in K_1 \cup K_2 : Tx = x\} \ne \phi$,then F(T) consists of a single point.

Proof:

Assume the contrary that Let $z,w \in K_1 \cup K_2$ be two distinct fixed points of T.Then

$$\|z-w\| = \|Tz - Tw\| \le p\|z-w\| + q\{\|z-Tz\| + \|w-Tw\|\} + r\{\|z-Tw\| + \|Tz-w\|\} + s\{\frac{\|z-Tw\| + \|w-Tz\|}{2}\}$$

$$= (p+2q+2(s/2))\|z-w\|$$

Implies z = w,since p+2q+2(s/2)<1.

Hence the proof.

Theorem 3.2

Let K_1 and K_2 be closed subset of a Banach space X.An operator $T: K_1 \cup K_2 \to K_1 \cup K_2$ *with* $T(K_1) \subset K_2$ *and* $T(K_2) \subset K_1$ and satisfies (2.4) with $0 \le p,q,r,s < 1$ *and* $p+2q+2r+2(s/2) < 1$. Then the sequence $\{x_n\}$ in $K_1 \cup K_2$ satisfies $\lim_{n\to\infty}(x_n - Tx_n) = 0$ and the sequence $\{x_n\}$ converges to the unique fixed point of T.

Proof:

Let $x_0 \in K_1$.

Define

$$x_n = Tx_{n-1} = T^n x_0 \; then \; \|Tx_n - Tx_m\| \le p\|x_n - x_m\| + q\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\} + r\{\|x_n - Tx_m\| + \|Tx_n - x_m\|\} + s\{\frac{\|x_n - Tx_m\| + \|Tx_n - x_m\|}{2}\}$$

By the triangle inequality we have

$$\|Tx_n - Tx_m\| \le p\{\|x_n - Tx_n\| + \|Tx_n - Tx_m\| + \|x_m - Tx_m\|\} + q\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\} + r\{\|x_n - Tx_n\| + \|x_m - Tx_m\| + 2\|Tx_n - Tx_m\|\} + s\{\frac{\|x_n - Tx_n\| + \|x_m - Tx_m\| + 2\|Tx_n - Tx_m\|}{2}\}$$

$$\Rightarrow \|Tx_n - Tx_m\| \le (\frac{p+q+r+(s/2)}{1-p-2r-2(s/2)})\{\|x_n - Tx_m\| + \|x_m - Tx_m\|\}$$

Observe from the hypothesis that ,the right hand side of the inequality tends to zero as $n \to \infty$ Hence $\{Tx_n\}$ is a Cauchy sequence .

Since $K_1 \cup K_2$ is complete,then it converges to limit ,say $z \in K_1 \cup K_2$ .that is $\lim_{n\to\infty} Tx_n = z$. Note that the sub sequence $\{x_{2n}\} \in K_1$ and the sub sequence $\{x_{2n+1}\} \in K_2$ thus $z \in K_1 \cap K_2 \ne \phi$.

Then we apply the triangle inequality and the fact that q<1 to get

$$\begin{aligned}
\|z - Tz\| \le \|Tz - Tx_n\| \le\ & p\|z - x_n\| + q\{\|x_n - Tx_n\| + \|z - Tz\|\} + \\
& r\{\|x_n - Tz\| + \|Tx_n - z\|\} + \\
& s\{\frac{\|x_n - Tz\| + \|Tx_n - z\|}{2}\} \\
\le\ & p\|z - x_n\| + q\{\|x_n - Tx_n\| + \|z - Tz\|\} + \\
& r\{\|z - x_n\| + \|z - Tz\| + \|x_n - Tx_n\| + \|z - x_n\|\} + \\
& s\{\frac{\|z - x_n\| + \|z - Tz\| + \|x_n - Tx_n\| + \|z - x_n\|}{2}\} \\
\le\ & (\frac{p+2r+2(s/2)}{1-q-r-(s/2)})\|z - x_n\| + (\frac{q+r+(s/2)}{1-q-r-(s/2)})\|x_n - Tx_n\|
\end{aligned}$$

It follows from $\lim_{n\to\infty} x_n = z \, and \, \lim_{n\to\infty}(Tx_n - x_n) = 0$

Implies that z is the fixed point of T which is unique by the proposition 3.1.

Hence the proof.

Corollary 3.3

Let K_1 and K_2 be closed subset of a Banach space X.An operator $T: K_1 \cup K_2 \to K_1 \cup K_2 \, with \, T(K_1) \subset K_2 \, and \, T(K_2) \subset K_1$ and satisfies (2.3) with $0 \le p, q < 1 and\ p + 2q < 1$. Then the sequence $\{x_n\}$ in $K_1 \cup K_2$ satisfies $\lim_{n\to\infty}(x_n - Tx_n) = 0$ and the sequence $\{x_n\}$ converges to the unique fixed point of T.

Proof:

The proof of corollary follows immediate ,by taking r=0 and s=0 in the above theorem.

Corollary 3.4

Let $\{K_i, i = 1,2,..........m\}$ be non - empty closed subsets of a Banach Space X and let $T: \bigcup_{i=1}^m K_i \to \bigcup_{i=1}^m K_i$ satisfies the following conditions:

1. $T(K_i) \subseteq K_{i+1}\ \ for\, 1 \le i \le m\ and\ K_{m+1} = K_1.$
2. There exits $0 \le p, q < 1\ such\ that$
$$\|Tx - Ty\| \le p\|x - y\| + q\{\|Tx - x\| + \|Ty - y\|\}\ \ for\ \ all\ \ x \in K_i, y \in K_{i+1}\ \ and\ \ 1 \le i \le m.$$

Proof:

It is sufficient to prove that for a given $x \in \bigcup_{i=1}^m K_i$,infinitely many terms of the sequence $T^n x$ lie in each K_i. Thus $\bigcap_{i=1}^m K_i \ne \phi.$ Then the operator $T: \bigcap_{i=1}^m K_i \to \bigcap_{i=1}^m K_i$ satisfies the conditions of the theorem 1 in [11].

Example 3.5

Let $K_1 = [0, \frac{1}{2}], K_2 = [\frac{1}{4}, \frac{3}{4}] \, and \, K_3 = [\frac{1}{6}, 1]$ Define the operator T as follows :

$$Tx = \begin{cases} \frac{2}{5} & if \; 0 \le x \le \frac{1}{2} \\ \frac{2}{3}(1-x) & if \; \frac{1}{2} < x \le 1 \end{cases}$$

Observe that $T(K_1) = \left\{\frac{2}{5}\right\} \subset K_2, T(K_2) = \left[\frac{1}{6}, \frac{2}{5}\right] \subset K_3, T(K_3) = \left[0, \frac{2}{5}\right] \subset K_1$

We have shown in Example 2.4 that this operator T is in the class $D\left(\frac{1}{4}, \frac{1}{4}\right)$ and has a unique fixed point.

If we impose an additional condition on the operator ,more precisely on the constants p,q,r and s we get the following theorem.

Theorem 3.6:

Let K_1 and K_2 be closed subset of a Banach space X.An operator $T : K_1 \cup K_2 \to K_1 \cup K_2 \; with \, T(K_1) \subset K_2 \; and \, T(K_2) \subset K_1$ and satisfies (2.4) with $0 \le p, q, r, 2(s/2) < 1 and \; p + 2q + 2r + 2(s/2) < 1.$ Then

a) T has a unique fixed point P in $K_1 \cap K_2$.

b) $\|Tx - P\| < \|x - P\| \;\; for \, all \;\; x \in K_1 \cup K_2$, where P is the fixed point of T.

Proof:

a.Take a point $x_0 \in K$.

$Define \quad x_{n+1} = Tx_n \quad for \, n = 0,1,2........ then \; we \; have$

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| = \|Tx_n - Tx_{n+1}\| \le p\|x_n - x_{n+1}\| + \\ q\{\|x_n - Tx_n\| + \|x_{n+1} - Tx_{n+1}\|\} + \\ r\{\|x_n - Tx_{n+1}\| + \|Tx_n - x_{n+1}\|\} \\ + s\{\frac{\|x_n - Tx_{n+1}\| + \|Tx_n - x_{n+1}\|}{2}\} \end{aligned}$$

This inequality implies

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| \le \left(\frac{p+q+r+s/2}{1-q-r-s/2}\right)\|x_n - Tx_n\| \le ...... \le \\ \left(\frac{p+q+r+s/2}{1-q-r-s/2}\right)^n \|x_0 - Tx_0\|. \end{aligned}$$

Hence we obtain

$\|x_n - Tx_n\| \to 0 \; as \; n \to \infty,$ implying that $\lim_{n\to\infty} x_n = P,$ where P is the fixed point of T.

Since the sub sequence $\{x_{2n}\} \in K_1$,and the sub sequence $\{x_{2n+1}\} \in K_{2,} \; then \; P \in K_1 \cap K_2.$ The uniqueness follows from proposition 3.1.

b.Let P be the fixed point of T and $x \in K_1 \cup K_2$.

Then using (2.4) and the triangle inequality ,we have

$$\|Tx-P\| \le \|Tx-TP\|+\|TP-P\|$$
$$\le p\|x-P\|+q\{\|x-Tx\|+\|P-TP\|\}+$$
$$r\{\|x-TP\|+\|Tx-P\|\}+s\left\{\frac{\|x-TP\|+\|Tx-P\|}{2}\right\}$$
$$\le p\|x-P\|+q\{\|x-P\|+\|P-Tx\|\}+$$
$$r\{\|x-P\|+\|Tx-P\|\}+s\left\{\frac{\|x-P\|+\|Tx-P\|}{2}\right\}$$

This inequality implies

$\|Tx-P\| \le \left(\frac{p+q+r+s/2}{1-q-r-s/2}\right)\|x-P\| < \|x-P\|$ as $\left(\frac{p+q+r+s/2}{1-q-r-s/2}\right)<1$,

Which completes the proof.

Corollary 3.7

Let K_1 and K_2 be closed subsets of a Banach Space X.suppose that the operator $T:K_1\cup K_2 \to K_1\cup K_2$ *with* $T(K_1)\subset K_2$ *and* $T(K_2)\subset K_1$ satisfies (2.3) with $0\le p,q,r<1$ *and* $p+2q+2r<1$.Then

a) T has a unique fixed point P in $K_1 \cap K_2$.

b) $\|Tx-P\|<\|x-P\|$ *for all* $x\in K_1\bigcup K_2$, where P is the fixed point of T.

Proof:

The proof of corollary follows immediate ,by taking s=0 in the above theorem.

3. CONCLUSION

We have proved some fixed point theorems for cyclic mapping with contraction and expansive conditions in Banach Spaces.The presented results generalize the results proved in various spaces and extend some results from the literature.

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