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Common Fixed Point Theorems For Weakly Compatible Mappings In Generalized Fuzzy Metric Spaces

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Abstract: In this paper, we prove common fixed-point theorems for weakly compatible mappings satisfying common E.A. Like property in generalized fuzzy metric space. We generalize the result of Arihan Jain et al using rational inequality.

Keywords: Generalized Fuzzy Metric Space, Common E.A. Like Property, Compatible Maps, Weakly Compatible Maps.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 1965, Zadeh [15] introduced the concept of fuzzy set. Following the concept of fuzzy sets Kramosil and Michalek [6] introduced the concept of fuzzy metric space in 1975. George and Veeramani [2] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. In 2006, Sedghi and Shobe [12] defined a new notion called M- fuzzy metric spaces and proved a common fixed point theorem for four weakly compatible mappings in this space. Recently, Jain et al. [11] improved the result of Kumar and Pant [8] by dropping the condition of continuity of the mapping and using semi and weak compatibility of the mapping in place of compatibility.

In this paper, we prove a common fixed point theorem for weakly compatible mappings satisfying common E.A. Like property in generalized fuzzy metric space, which generalize the result of Jain at al. [11] using rational inequality.

2. PRELIMINARIES

Definition 2.1

A 3-tuple (X, $\mathcal{M}$, $*$) is called $\mathcal{M}$-fuzzy metric space (generalized fuzzy metric space) if X is an arbitrary non empty set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions: for each x, y, z, a $\in$ X and t, s > 0.

(i) $\mathcal{M}(x, y, z, t) > 0$,

(ii) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(iii) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function.

(iv) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(iv) $\mathcal{M}(x, y, z, .) : [0, \infty) \to [0,1]$ is left continuous,

(v) $\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$ for all x, y, z $\in$ X.

Definition: 2.2

Let (X, M, *) be an M-fuzzy metric space and $\{x_n\}$ be a sequence in X

i) A sequence $\{x_n\}$ in S is said to be convergent to a point $x \in X$, (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x, x, x_n, t) = 1$ for all $t > 0$

ii) A sequence$\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

iii) A M-fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition :2.3

A function $\mathcal{M}$ is continuous in $\mathcal{M}$- fuzzy metric space if and only if whenever $x_n \to x$, $y_n \to y$ and $z_n \to z$, then $\lim_{n\to\infty} M(x_n, y_n, z_n, t) = M(x, y, z, t)$ for all $t > 0$.

Definition: 2.4

Let A and B be mappings from $\mathcal{M}$ - fuzzy metric space (X, $\mathcal{M}$,*) into itself. The

mappings A and B are said to be weakly compatible if they commute at their coincidence

points, i.e. Ax = Bx implies ABx = BAx.

Definition : 2.5

Suppose A and S be two maps from a $\mathcal{M}$ -fuzzy metric space(X, $\mathcal{M}$,*) into itself. Then they are said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Lemma : 2.6

Let $\{x_n\}$ be a sequence in a $\mathcal{M}$ -fuzzy metric space $(X, \mathcal{M},*)$ with (FM-6). If there exists a number $h > 1$ such that $M(x_{n+1}, x_n, x_n, ht) \leq M(x_{n+2},x_{n+1}, x_{n+1}, t)$ for all $t > 0$ and

$n = 1,2\ldots$ .Then $\{x_n\}$ is Cauchy sequence in X.

Lemma: 2.7

If for all $x, y,z \in X$, $t > 0$ and for a number $h > 1$, $\mathcal{M}(x, y, z, ht) \leq \mathcal{M}(x, y, z,t)$ then

$x = y= z$..

3. Main results

Theorem 3.1:

Let $(X, \mathcal{M},*)$ be a complete generalized fuzzy metric space where * is continuous t-norm and satisfies $t * t \geq t$ for all $t\in [0, 1]$. Let A, B, S and T be self mappings of a generalized fuzzy metric space satisfying the following conditions:

(3.1.1) For all $x, y, z\in X$, $t > 0$ and $h > 1$.

$$\mathcal{M}(Ax,By,Bz,ht) \leq \min\{\mathcal{M}(Sx,Ax,Ay,t),\mathcal{M}(Ty,By,Bz,t),\frac{r\mathcal{M}(Sx, By, Bz,t)+s\mathcal{M}(Sx,Ty,Tz,t)}{r\mathcal{M}(By,Ty,Bz,t)+s}\},$$

where $r,s \geq 0$ with r and s cannot be simultaneously 0.

(3.1.2) Pairs (A,S) and (B,T) satisfy common E.A. Like property.

(3.1.3) Pairs (A,S) and (B,T)are weakly compatible.

Then A, B, S and T have a unique common fixed point in X.

Proof:

Since (A,S) and (B,T) satisfy common E.A. Like property, therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = w$, where $w \in S(X)\cap T(X)$ or $z \in A(X)\cap B(X)$.

Suppose $z \in S(X)\cap T(X)$, now we have $\lim_{n\to\infty} Ax_n = w \in S(X)$ then $w = Su$ for some $u \in X$.

Now, we claim that $Au = Su$, form (3.1.1) we have,

$$\mathcal{M}(Au,By_n,By_n,ht) \leq \min\{\mathcal{M}(Su,Au,Ay_n,t), \mathcal{M}(Ty_n,By_n,By_n,t),$$

$$\frac{r\mathcal{M}(Su, By_n, By_n,t) + s\mathcal{M}(Su,Ty_n,Ty_n,t)}{r\mathcal{M}(By_n,Ty_n,By_n,t) + s}\}.$$

Taking limit $n \to\infty$, we get

$$\mathcal{M}(Au,By_n,By_n, ht) \leq \min\{\mathcal{M}(w,Au,w,t), \mathcal{M}(w,w,w,t),\frac{r\mathcal{M}(w,w,w,t) + s\mathcal{M}(w,w,w,t)}{r\mathcal{M}(w,w,w,t) + s}\}$$

$\mathcal{M}(Au,w,w,ht) \le \min\{\mathcal{M}(w,Au,w,t),1,1\}$

$\mathcal{M}(Au,w,w,ht) \le \mathcal{M}(w,Au,w,t)$

$\mathcal{M}(Au,w,w,ht) \le \mathcal{M}(Au,w,w,t)$.

Lemma (2.7) implies that Au = w = Su.

Since the pair (A,S) is weak compatible, therefore Aw = ASu = SAu= Sw.

Again, $\lim_{n\to\infty} By_n = w \in T(X)$ then w = Tv for some $v \in X$.

Now, we claim that Tv = Bv, from (3.1.1) we have,

$\mathcal{M}(Ax_n,Bv,Bv,ht) \le \min\{\mathcal{M}(Sx_n,Ax_n,Av,t),\mathcal{M}(Tv,Bv,Bv,t),$

$\frac{r\mathcal{M}(Sx_n,\ Bv,\ Bv,t)\ +\ s\mathcal{M}(Sx_n,Tv,Tv,t)}{r\mathcal{M}(Bv,Tv,Bv,t)+\ s}\}$.

Taking limit $n \to\infty$, we get

$\mathcal{M}(w,Bv,Bv,ht) \le \min\{\mathcal{M}(w,w,Av,t),\ \mathcal{M}(w,Bv,Bv,t),\frac{\mathcal{M}(w,\ Bv,\ Bv,t)\ +\ \mathcal{M}(w,w,w,t)}{\mathcal{M}(Bv,w,Bv,t)+\ s}\}$,

$\mathcal{M}(w,Bv,Bv,ht) \le \min\{1,\mathcal{M}(w,Bv,Bv,t),1\}$

$\mathcal{M}(w,Bv,Bv,ht) \le \mathcal{M}(w,Bv,Bv,t)$

$\mathcal{M}(Bv,Bv,w,ht) \le \mathcal{M}(Bv,Bv,w,t)$

Lemma (2.7) implies that Bv = w = Tv = Av.

Since the pair (B,T) is weak compatible, therefore Tw = TBv = BTv = Bw.

Now, we show that Aw = w, from (3.1.1) we have,

$\mathcal{M}(Aw,By_n,y_n,ht) \le \min\{\mathcal{M}(Sw,Aw,Ay_n,t),\ \mathcal{M}(Ty_n,By_n,By_n,t),$

$\frac{r\mathcal{M}(Sw,\ By_n,\ By_n,t)+\ s\mathcal{M}(Sz,Ty_n,Ty_n,t)}{r\mathcal{M}(By_n,Ty_n,By_n,t)\ +\ s}\}$.

Taking limit $n \to\infty$, we get

$\mathcal{M}(Aw,w,w,ht) \le \min\{\mathcal{M}(Aw,Aw,Aw,t),\ \mathcal{M}(w,w,w,t),\frac{r\mathcal{M}(Aw,w,\ w,t)\ +\ s\ \mathcal{M}(Aw,w,w,t)}{r\mathcal{M}(w,w,w,t)+\ s}\}$

$\mathcal{M}(Aw,w,w,ht) \le \min\{1,1,\frac{r\ \mathcal{M}(Aw,w,\ w,t)}{r\ \mathcal{M}(w,w,w,t)+s}\}$

$\mathcal{M}(Aw,w,w,ht) \le \mathcal{M}(Aw,w,w,t)$.

Lemma (2.7) implies that Aw = w.

Now, we show that Bw = w, from (3.1.1) we have,

$\mathcal{M}(Ax_n,Bw,Bw,ht) \le \min\{\mathcal{M}(Sx_n,Ax_n,Aw,t),\ \mathcal{M}(Tw,Bw,Bw,t),$

$\frac{r\mathcal{M}(Sx_n, Bw, Bw,t)+ s \mathcal{M}(Sx_n,Tw,Tw,t)}{r \mathcal{M}(By,Tw,Bw,t)+ s}\}$.

Taking limit n $\to\infty$, we get

$\mathcal{M}(w,Bw,Bw,ht) \leq \min\{\mathcal{M}(w,w,w,t),\mathcal{M}(Bw,Bw,Bw,t),\frac{r\mathcal{M}(w, Bw, Bw,t)+ s \mathcal{M}(w,Bw,Tw,t)}{r\mathcal{M}(Bw,Bw,Bw,t) + s}\}$

$\mathcal{M}(w,Bw,Bw,ht) \leq \min\{1,1, \mathcal{M}(w, Bw, Bw, t)$

$\mathcal{M}(w,Bw,Bw,ht) \leq\mathcal{M}(w, Bw, Bw, t)$

$\mathcal{M}(Bw,Bw, w, ht) \leq\mathcal{M}(Bw, Bw, w, t)$

Lemma (2.7) implies that Bw = w.

Hence, Aw = Sw = Bw = Tw = w.

Thus w is a common fixed point of A,B,S and T

To prove uniqueness we suppose that p and q are two common fixed point of A,B,S and T such that p $\neq$ q, then from (3.1.1) we have,

$\mathcal{M}(Ap,Bq,Bq,ht) \leq \min\{\mathcal{M}(Sp,Ap,Aq,t),\mathcal{M}(Tq,Bq,Bq,t),$

$\frac{r\mathcal{M}(Sp, Bq, Bq,t)+s\mathcal{M}(Sp,Tq,Tq,t)}{r\mathcal{M}(Bq,Tq,Bq,t)+s}\}$

$\mathcal{M}(p,q,q,ht) \leq \min\{\mathcal{M}(p,p,q,t),\mathcal{M}(q,q,q,t),\frac{r\mathcal{M}(p,q,q,t)+ s \mathcal{M}(p,q,q,t)}{r \mathcal{M}(q,q,q,t) + s}\}$

$\mathcal{M}(p,q,q,ht) \leq \min\{1,1, \mathcal{M}(p,q,q,t)$

$\mathcal{M}(p,q,q,ht) \leq\mathcal{M}(p,q,q,t)$

Lemma (2.8) implies that p = q.

Corollary: 3.2

Let (X, $\mathcal{M}$,*) be a complete generalized fuzzy metric space where * is continuous t-norm and satisfies t * t $\geq$ t for all t$\in$ [0, 1]. Let A, B, S and T be self mappings of a generalized fuzzy metric space satisfying the following conditions:

(3.2.1) For all x, y, z $\in$ X , t > 0 and h > 1.

$\mathcal{M}(Ax, By, Bz, ht) \leq \min\{\mathcal{M}(Sx, Ax,Ay, t),\mathcal{M}(Ty, By, Bz,t),\frac{r\mathcal{M}(Sx, By, Bz,t)+s\mathcal{M}(Sx,Ty,Tz,t)}{r\mathcal{M}(By,Ty,Bz,t)+s}\}$,

where r,s $\geq$ 0 with r and s cannot be simultaneously 0.

(3.2.2) Pairs (A, S) and (B, T) satisfy common E.A. Like property.

(3.2.3) Pairs (A, S) and (B, T) are semi- compatible.

Then A, B, S and T have a unique common fixed point in X.

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