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A note on harmonic functions

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Abstract

Every function u regular and harmonic in a region G possesses for every sphere S-centre P(x, yz) and radius R-lying entirely inside G the mean value property.

$$\frac{1}{4\pi R^2}\left[\int\int_s u(Q)\,dS_Q - u(p)\right] \text{ -------------- } \textbf{(1)}$$

The converse of this thereon, that if U is continuous and possesses the property (1) for every sphere is G then U is harmonic in G, was discovered by Bocher and Koebe. In other words, the property (1) is a characteristic one for functions harmonic in G. A number of conditions each of which characterises a harmonic function have since been given for, instance Zaremba proved.

i.e. $\lim_{h \to o} \frac{1}{h^2}\left[\sum_{x,y,z}\{u(x+h,y,z)+u(x-h,y,z)\}-64(xyz)\right]=0$

***Key words*: Poissons equation, pockets equation, saks's theorem 'n' dimensions, greens theorem, Harmonic function.**

1. INTRODUCTION:

According to a well known theorem every function u regular and harmonic in a region G possesses for every sphere S-centre P(x, yz) and radius R-lying entirely inside G the mean value property.

$$\frac{1}{4\pi R^2}\left[\int\int_s u(Q)\,dS_Q - u(p)\right] \text{ -------------- (1)}$$

The converse of this thereon, that if U is continuous and possesses the property (1) for every sphere is G then U is harmonic in G, was discovered by Bocher and Koebe. In other words, the property (1) is a characteristic one for functions harmonic in G. A number of conditions each of which characterises a harmonic function have since been given for, instance Zaremba proved that if

$$\lim_{h \to o} \frac{1}{h^2}\left[\sum_{x,y,z}\{u(x+h,y,z)+u(x-h,y,z)\}-64(xyz)\right]=0$$

then $\nabla^2 u=0$, the condition (1) was generalised by Blaske[2] and may be stated as follows. A necessary and sufficient condition that the continuous function U be harmonic in G is

$$\lim_{R \to o} \frac{1}{R^2}\left[\frac{1}{4\pi R^2}\iint_s u(Q)\,dS_Q - u(p)\right] = 0$$

for every sphere S in G. Results suitable to those G. Zarcmba and Blaschke have been recently obtained by Kappos[2]. Saks[4] has used Blaschke's theorem to derive two other properties either of which may be regarded as the defining property of harmonic functions. in this paper it will be shown that results analogoes to those of Blaschke and saks may be obtained for solutions of poisson's equation and also of the equation $\nabla^2 u + Cu = 0$. The following results are obtained.

I) if $\displaystyle \lim_{R \to o} \frac{1}{R^2}\frac{1}{4\pi R^2}\iint_s uds - u(p) + \iiint_{q_k}\left(\frac{1}{r}-\frac{1}{R}\right)\mu dr = 0$

then $\nabla^2 u + 4\pi\mu = 0$

II) if $\displaystyle \lim_{R \to o} \frac{1}{R^2}\left[\frac{1}{4\pi R^2}\iint_s uds - u(p)\,\frac{f(R)}{f(0)}\right] = 0$

then $\displaystyle \nabla^2 u - 3\,\frac{f^{11}(0)}{f(0)}\,u = 0$

The first reduces to Blaschke's Théorem for $\mu = 0$ and the second is the convers of a mean value theorem due to weber[5]. Making the same use of theorem I and II as saks has done of Blaschke's theorem, the following further results are obtained.

III. If $\displaystyle \lim_{R \to o} \frac{1}{2R}\left[\frac{1}{4\pi}\iint_s \frac{\partial U}{\partial R}dQ + \frac{1}{R^2}\iiint_k v\mu dr\right] = 0$

then $\nabla^2 u + 4\pi\mu = 0$

IV. Let $\nabla^2 V = 0$ and $V \neq 0$, Then if

If $\displaystyle \lim_{R \to o} \frac{1}{R^3}\left[\frac{1}{4\pi}\iint_s\left(V\frac{\partial U}{\partial R} - u\frac{\partial U}{\partial R}\right)ds + \iiint_k \mu dr\right] = 0$

then $\nabla^2 u + 4\pi\mu = 0$

V. If $\displaystyle \lim_{R \to o} \frac{1}{4\pi R}\iint_s \frac{\partial u}{\partial R}\,d\Omega = cu(p)$ then $\nabla^2 u - 3c\mu = 0$

The theorems III and IV correspond to those of saks, to which they reduce when μ = 0/ The restrictions imposed on the functions, u, μ,f in the theorems are mentioned below, in the proper contexts.

2. Lemma

Let P(x, y, z) be an interior point of G and let the function Ψ (x, y, z, r) = Ψ (p;r) satisfy the conditions : (I) Ψ (x, y, z, r) is continuous in all its, arguments, and (II) $\lim_{r \to o}$ (x, y, z, r) = 0, Then if

$$J(P) = \frac{1}{a^3}\int_0^a r^4\,\Psi(p:r)\,dr$$

$$\lim_{P^1 \to P} \frac{J(P^1) - J(P)}{P^1 P} = 0$$

In fact we may choose axes so that pp^1 is parallel to the axis. Let (x, y, z + h) be the co-ordinates of P^1

Then – $\frac{J(P^1)-J(P)}{P^1P} = \frac{1}{a^3}\int_0^a \Psi \frac{(P^1:r)-4(p:r)}{h} r^u dr$

Since Ψ (P ; r) → 0 as r → 0. We can find r_o such that 1Ψ (p : r) | $\langle \frac{h^2}{2}$ for r < r_o and since Ψ (p : r) = Ψ (x, y, z, r) is continuous in all its arguments continuity will be uniform. then r_o will depend upon h only r_o Ξ r_o (h), and the above inequality will hold also for Ψ (p' : r) with same r_o. Since the upper limit a in J(P) is arbitrary we may choose it so hat a <r_o (h). we then have

$$\left|\frac{J(P^1)-J(P)}{P^1P}\right| = \left|\frac{1}{a^3}\int_0^a \Psi \frac{(P^1:r)-\Psi(p:r)}{h} r^u dr\right| < \frac{h}{a^3}\int_0^a r^4 dr = \frac{1}{5}ha^2$$

from which it is evident that $\lim_{P^1 \to p} \frac{J(P^1)-J(P)}{p^1p} = 0$

This proves the Lemma. i am not aware of the proof given by Blaschke of his theorem, but it must be noticed that his theorem is an immediate consequence of our Lemma. For writing,

$$\Psi\,(P:R) = \frac{1}{R^2}\left[\frac{1}{4\pi R^2}\iint_s u(Q)\,dS_Q - u(p)\right]$$

so that if u is continuous, Ψ satisfies the conditions of the Lemma.

We then have u (P) = $\frac{1}{4\pi R^2}\iint_s u(Q)\,ds_Q - R^2\Psi(P:R)$ and integrating from O to a w.r.t. R, we get

$$u(P) = \frac{3}{4\pi a^3}\iint_k udr - \frac{3}{a^3}\int_0^a R^4\Psi(p;R)dR$$

(K is the volume inside a sphere of rad, a and centre P.) = I(P) – J (P)

To calculate $\frac{\partial up}{\partial Z}$ for instance, we take P^1(x, y, z + h) and compute the limit $\lim_{h \to o} \frac{u(P^1)-u(P)}{h}$

The Lemma shows that J(P) disappears in this process and we have

$$\frac{\partial u}{\partial Z} = \lim_{h \to o} \frac{I(P^1)-I(P)}{h}$$

From this point onwards the arguments runs on the usual lines and we infer the existence and continuity of the second derivatives of u. We now have by greens theorem

$$\frac{1}{4\pi R^2}\iint_s u(Q)\,ds_Q - u(P) = \frac{1}{4\pi}\iiint_k \left(\frac{1}{r}-\frac{1}{R}\right)\nabla^2 udr$$

where r = distance of dr from P, the centre of the sphere-

$$\therefore \lim_{R \to o} \frac{1}{R^2}\left[\frac{1}{4\pi R^2}\iint_s u(Q)\,dS_Q - u(p)\right]$$

$$= \lim_{R \to o} \frac{1}{4\pi R^2} \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) \nabla^2 u dr$$

$$= (\nabla^2 u)p. \lim_{R \to o} \frac{1}{4\pi R^2} \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) dr = \frac{1}{6} (\nabla^2 u)_p.$$

Since the value of the left hand side is zero, it follows that ∇^2 u = 0. This proves the sufficiency of the condition. Since bygauss's theorem the condition is evidently necessary we have herewith proved Blaschke's theorem.

3. Poisson's equation

Let μ be a given function continuously differentiable in G. Let u be continuous in G and let it satisfy, further, for every sphere in G, the condition

$$\lim_{R \to o} \frac{1}{R^2} \left[\frac{1}{4\pi R^2} \iint_s U(Q)\, dsQ - u(P) + \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) \mu dr \right] = 0 \qquad (3.1)$$

put

$$\Psi\,(P:R) = \frac{1}{R^2} \left[\frac{1}{4\pi R^2} \iint_s U(Q)\, dsQ - u(P) + \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) \mu dr \right] = 0$$

so that, $u\,(P) = \frac{1}{4\pi R^2} \iint_s U(Q)\, dsQ + \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) \mu dr - R\Psi(P;R)$

Multifying by R^2 and integrating from O to R w.r.t. R. this gives

$$u(p) = \frac{3}{4\pi R^3} \iiint_k u dr + \iiint_k \frac{(R - r^2)(2R + r)}{2rR^3} \mu dr - \frac{3}{R^3} \int_O^R R^4 \Psi(P:R)\, dR$$

= I (P) + L (P) – J (P)

Taking P^1 to be the point (x, y, z + h), we compute as before

$$\frac{\partial up}{\partial Z} = \lim_{h \to o} \frac{u(P^1) - u(P)}{h}.$$

In this process the integral J(P) will disappear by the Lemma. If μ is continuously differentiable, then $\mu_1 = \frac{(R - r)^2 (2R + r)}{2R^3}$ μ is also continuous and continuously differentiable and by a well known theorem in potential theory L(P) = $\iiint_k \frac{\mu_1}{r}$ dr possesses continuous second derivatives. We may now apply the usual type of reasoning to I(P) and infer the existence of continuous second derivatives for I(P). Thus we conclude that under the given condition u is twice continuously differentiable. But if this is so we have

$$\lim_{R \to o} = \frac{1}{R^2} \left[\frac{1}{4\pi R^2} \iint_s U(Q)\, dsQ - u(P) \right]$$

$$= \lim_{R \to o} \frac{1}{R^2} \frac{1}{4\pi} \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) \nabla^2 u dr$$

$$(\nabla^2 u) \lim_{R \to o} \frac{1}{4\pi R^2} \iiint_k \left(\frac{1}{r} - \frac{1}{R}\right) dr$$

$= \frac{1}{6}(\nabla^2 u)$

while $\lim_{R \to o} \frac{1}{R^2} \iiint \left(\frac{1}{r} - \frac{1}{R}\right) \mu dr = \mu p \lim_{R \to o} \frac{1}{R^2} \iiint_K \left(\frac{1}{r} - \frac{1}{R}\right) dr = \frac{4\pi}{6} \mu p$

Hence the given condition is equivalent to ∇^2u + 4πμ = 0
We have thus the result:

If u is continuous in G and satisfies the condition (3.1) for every sphere in G, then ∇^2u + 4πμ=o. For μ = 0, this reduces to Blaschke theorem, The condition (3.1) will be satisfied in particular if for every sphere in G.

$$u(P) = \frac{1}{4\pi R^2} \iint_s u(Q) dsq + \iiint_K \left(\frac{1}{r} - \frac{1}{R}\right) \mu dr \qquad (3.2)$$

Thus if the continuous function u satisfies the condition (3.2) for every sphere in G, then u satisfies poisson's equation at every point of G. This is an extension to poisson's equation of the converse theorem of Bocher and Koche and is well known[7].

4. Pockets equation

Next consider the equation ∇^2u + cu = 0. Every regular solution of this equation satisfies the mean value property.

$$\frac{1}{4\pi R^2} \iint_s u(Q) ds_Q = u(P) \frac{Sin R\sqrt{C}}{R\sqrt{C}}$$

4.1

a result due to weber, we may prove a converse theorem in this case similar to that of Blaschke.

Let f(x) be twice continuous differentiable with

f (O) finite, ≠ O and f' (O) = O. Let the continuous function u satisfy the condition

$$\lim_{R \to o} \Psi(p;R) = O$$

where $\Psi (P;R) = \frac{1}{R^2} \left[\frac{1}{4\pi R^2} \iint_s U(Q) dsQ - u(P) \frac{f(R)}{f(0)} \right]$

we then have

$$\frac{f(R)}{f(0)} u(P) = \frac{1}{4\pi R^2} \iint_s U(Q) ds_Q - R^2 \Psi(p;R) \quad =$$

multiplying by R^2 and integrating firm O to a w.r.t. R. we get

$$Cu(P) = \frac{1}{4\pi} \iiint_K udr - \int_o^a R^4 \Psi(P;R) dR$$

with $C = \int_o^a \frac{f(R)}{f(0)} R^2 dR \quad =$

By reasoning exactly as before we conclude that U(P) is twice continuously differentiable

Now $\frac{1}{R^2}\left[\frac{1}{4\pi R^2}\int\int_s U(Q)\, dsQ - u(P)\frac{f(R)}{f(0)}\right]$

$$= \frac{1}{R^2}\left[\frac{1}{4\pi R^2}\int\int_s U(Q)\, ds_Q - u(P)\right] - \frac{u(P)}{f(o)}.\frac{f(R)-(0)}{R^2}$$

Since u is twice continuously differentiable

$$= \lim_{R\to o} \frac{1}{R}\left[\frac{1}{4\pi R^2}\int\int_s U(Q)\, ds_Q - u(P)\right] = \frac{1}{6}(\nabla^2 u)$$

and

$$\lim_{R\to o} \frac{f(R)-f(O)}{R^2} = \frac{1}{2} f^{11}(O)$$

Thus the assumed condition leads to $\nabla^2 u - \frac{3f^{11}(O)}{f(O)} - u = 0$

Obtained the following results

If u be continuous in G and if for every sphere in G

$$\lim_{R\to o} \frac{1}{R^2}\left[\frac{1}{4\pi R^2}\int\int_s U(Q)\, ds_Q - u(P)\frac{f(R)}{f(O)}\right] = 0 \qquad (4.2)$$

then $\nabla^2 u + cu = 0$ with $c = -\frac{3f^{11}(O)}{f(O)}$ provided that f(u) satisfies the conditions specified above.

In particular if $f^{11}(O)= 0$, then u must be harmonic. the condition (4.2) will be satisfied if for every sphere in G

$$\frac{1}{4\pi R^2}\int\int_s U(Q)\, ds_Q - u(P)\frac{f(R)}{f(O)}$$

Taking $f(x) = \frac{Sin\sqrt{C.x}}{\sqrt{C.x}}$ we obtain the converse of Weber's mean value theorem.

5. Generalisations to dimensions

These results may immediately be generalized to n dimensions. Thus taking the case of poisson's equation, the corresponding result will here be as follows: let u be continuous in G and let it satisfy for every sphere Q_R, centre P and radius R, in G, the condition.

$$\lim_{R\to o} \frac{1}{R^2}\left[\frac{1}{W_n R^{n-1}}\int \Omega R \int u(Q)d\Omega) - u(P) + \frac{1}{n-2}\int_{KR}\int \int\left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right)\mu dr\right] = 0 \qquad (5.1)$$

Where K_R is the volume inside ΩR, μ is a continuously differentiable function, and W_n is the surface area of the unit sphere is n dimensions. Then u is twice continuously differentiable and satisfies in G the equation.

$\nabla^2 u + W_n\mu=0$

$\nabla^2 \equiv \frac{\partial^2}{\partial xr^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$

In fact the proof of the differentiality of u proceeds exactly as before and then we have, by the corresponding form of greens theorem in n dimensions.

$$= \frac{1}{R^2}\left[\frac{1}{W_n R^{n-1}}\int_{\Omega R}....\int u\, d\Omega - u(P)\right] = \frac{1}{Wn(n-2)}\frac{1}{R^2}\iint_{KR}...\int \left(\frac{1}{r^{n-1}} - \frac{1}{R^{n-2}}\right)\nabla^2 \text{udr}$$

So that

$$\lim_{R\to o} \frac{1}{R^2}\left[\frac{1}{W_n R^{n-1}}\int_{\Omega R}....\int u\, d\Omega - u(P)\right]$$

$$= \frac{(\nabla^2 u)_p}{W_n(n-2)}.\lim_{R\to o}\frac{1}{R^2}\iint_{KR}....\int \left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right)dr$$

$$= \frac{(\nabla^2 u)_p}{Wn(n-2)} \lim_{R\to o} \frac{1}{R^2}\int_O^R \left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right)W_n\, r^{n-1}\, dr = \frac{1}{2n} (\nabla^2 \text{u})\text{P}$$

and $\lim_{R\to o}\frac{1}{R^2}\frac{1}{(n-2)}\iint_{KR}....\int \left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right)\mu dr = \frac{W_n}{2_n}\mu p$

Thus u satisfies the equation $\nabla^2 u + W_n u = 0$ again let u be continuous in G and satisfy the condition $\lim_{R\to o} \frac{1}{R^2}\left[\frac{1}{W_n R^{n-1}}\int_{\Omega R}....\int u\, d\Omega - u(P)\frac{f(R)}{f(0)}\right] = 0$ (5.2)

f(x) being such that $f(0) \neq 0$, $f'(0) = 0$, f^{11} (u) continuous
Then u satisfies the equation

$$\nabla^2 \text{u} \quad \frac{-nf^1(r)}{f(0)} \text{ u} = \text{O}.$$

For the differentiability of u being proved exactly as before we have

$$\lim_{R\to o}\frac{1}{R^2}\left[\frac{1}{W_n R^{n-1}}\int_{\Omega R}....\int u\, d\Omega - u(P)\right] = \frac{u(P)}{f(0)} \lim_{R\to o} \frac{f(R) - f(0)}{R^2}$$

i.e. $\frac{1}{2n}(\nabla^2 \text{u})\text{p} = \frac{1}{2}\frac{f''(0)}{f(0)}$ up

or $\nabla^2 \text{u} - nf^{11}(0)/f(0)$ (5.3)

The result will be true in particular if we replace the condition (5.2) by

$$\left[\frac{1}{W_n R^{n-1}}\int_{\Omega R}....\int u\, d\Omega - u(P).\frac{fu(P)}{f(0)}\right] \quad (5.4)$$

This result that under the condition (5.4) u satisfies the equation (5.3) has been proved as a particular case of a more general theorem and in a different way by H poritsky in a recent paper[8].

6. Analogues of saks's theorems

Consider first the case of poisson's equation. then the analogues of Saki's theorems to this case may be stated as follows:

1) Let u be continuous, with its first partial derivatives, in G and let it satisfy for every sphere in G the condition.

$$\lim_{R\to o}\frac{1}{2R}\left[\frac{1}{4\pi}\iint_s \frac{\partial u}{\partial R}d\Omega + \frac{1}{R^2}\iiint_K \mu dr\right] = 0 \quad (6.1)$$

Where μ is continuously differentiable. Then $\nabla^2 u+4\pi\mu=0$ (Here dΩ means the element of area on the unit sphere)

2) Let $\nabla^2 V=0$, V≠0 in G. If u be continuous with its partial drivatives in G and satisfies for every sphere in G, the condition.

$$\lim_{R \to o} \frac{1}{R^3}\left[\frac{1}{4\pi}\iint_s V\frac{\partial u}{\partial R}-u\frac{\partial v}{\partial R})ds+\iiint_K v\mu\, dr\right]=0 \qquad (6.2)$$

then u satisfies $\nabla^2 u+4\pi\mu=0$

The method of proof is identical with that used by Saks, except that where as saks makes use of Blaschke's theorem, we here use the analogue of Blaskchke's theorem proved in 3 above. from which the first theorem is an immediate consequence. following an ingenious procedure due to saks we may write

$$\Psi(R)=\frac{1}{R^2}\left[\frac{1}{4\pi R^2}\iint_s uds-u(p)+\iiint_K\left(\frac{1}{r}-\frac{1}{R}\right)\mu\, dr\right]$$

and apply L, Hospital's rule of the differential calculus to find $\lim_{R \to o}\Psi(R)$

This procedure gives

$$\lim_{R \to o}\Psi(R)=\lim_{R \to o}\frac{1}{2R}\frac{d}{dR}\left[\frac{1}{4\pi R^2}\iint_s uds-u(p)+\iiint_K\left(\frac{1}{r}-\frac{1}{R}\right)\mu\, dr\right]$$

$$\lim_{R \to o}\frac{1}{2R}\left[\frac{1}{4\pi}\iint_s\frac{\partial u}{\partial R}d\Omega+\frac{1}{R^2}+\iiint_k\mu dr\right]$$

If therefore the right – hand limit vanishes, so does $\lim_{R \to o}\Psi(R)$ and hence by the theorem of 3, u satisfies

$\nabla^2 u+4\pi\mu=0$. This proves theorem (1). The second theorem now follows from the first exactly as in Sak's paper. finally consider the analogue of Saks's theorem with respect to the equation $\nabla^2 u+cu=0$ let u be continuous, with its first partial derivatives is G. and let f(x) satisfy the conditions specified is 4.

If we now write, $\Psi(R)=\frac{1}{R^2}\left[\frac{1}{4\pi}\iint_s ud\Omega-u(P)\frac{f(R)}{f(0)}\right]$

we find as above

$$\lim_{R \to o}\Psi(R)=\lim_{R \to o}\frac{1}{2R}\left[\frac{1}{4\pi}\iint_s\frac{\partial u}{\partial R}d\Omega-u(P)\frac{f^1(R)}{f(0)}\right]$$

Under the hypothesis regarding f(x) we have

$$\lim_{R \to o}\frac{f^1(R)}{R}=\lim_{R \to o}\frac{f^1(R)-f^1(0)}{R}\ f^{11}(0)$$

we thus find that if

$$\lim_{R \to o}=\left[\frac{1}{4\pi R}\iint_s\frac{\partial u}{\partial R}d\Omega-u(P)\frac{f^{11}(0)}{f(0)}\right]=0$$

then $\lim_{R \to o} \Psi(R) = 0$ and hence by 4, u satisfies $\nabla^2 u - \frac{3 f^{11}(0)}{f(0)} u = 0$

Writing $C = \frac{f^{11}(r)}{f(0)}$ this result may be stated as follows:

If u be continuous with its first partial derivatives G and satisfies for every sphere in G the condition.

$$\lim_{R \to o} \frac{1}{4\pi R} \iint_s \frac{\partial u}{\partial R} d\Omega = C u(P) \qquad (6.3)$$

then u satisfies at every point of G the equation $\nabla^2 u - 3cu = 0$

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